$\boxed{\text{Hurwitz}}$

$C_1 \xrightarrow{\varphi} C_2$ then $2g_{C_1} - 2 \geq \deg\varphi\,(2 - g_{C_2}) + \underbrace{\sum_{P \in C_1} (e_\varphi(P) - 1)}_{\substack{\geq 0 \text{ so may} \\ \text{sum over fin pts.}}}$

$\boxed{\text{if } \varphi \text{ is separable}}$

<u>Proof</u>: Recall $2g_C - 2 = \deg(\operatorname{div}\omega)$

$\omega \in \Omega_C = \bar{k}(C)\,dx$ $[k(C) : k(x)]$ separable

• $C_1 \xrightarrow{\varphi} C_2 \longleftrightarrow \bar{k}(C_2) \overset{\varphi^*}{\hookrightarrow} \bar{k}(C_1)$ induces

<u>injection</u>

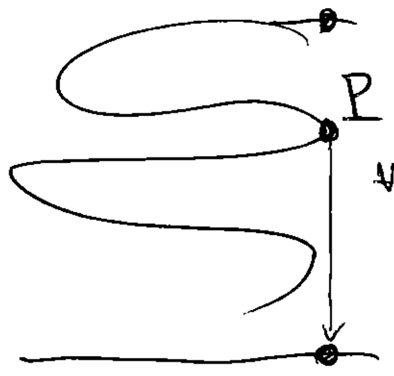
choose x

$k(x) \underset{sep}{—} k(C_2) \overset{sep}{—} k(C_1)$

$\Rightarrow dx$ basis also for Ω_{C_1}

$$\Omega_{C_2} \hookrightarrow \Omega_{C_1}$$

$$f\,dx \mapsto \varphi^*(f)\,d\varphi^*(x)$$

• 

Q

$\omega = f\,dt_Q$ in Q

$P \mapsto Q$ $\operatorname{ord}(\varphi^*(t_Q)) = u\,t_P^{e_\varphi(P)}$ $u \in \mathcal{O}_P^*$

$$\varphi^*(\omega) = \varphi^*(f)\,d\varphi^*(t_Q) = \varphi^*(f)\,d(u\,t^{e_\varphi(P)})$$

$$= \varphi^*(f)\Big[\underbrace{e_\varphi(P)\,u\,t^{e_\varphi(P)-1}}_{\substack{\operatorname{ord} = e_\varphi(P)-1 \\ \text{unless } e_\varphi(P) \equiv 0 \text{ in } \bar{k}}} + \underbrace{\frac{du}{dt_P}\,t^{e_\varphi(P)}}_{\operatorname{ord} \geq e_\varphi(P)}\Big)$$

So $\quad \operatorname{ord}_P \varphi^*(\omega) \geqslant \operatorname{ord}_P \varphi^*(f) + (e_\varphi(P) - 1)$

& $\quad \operatorname{ord}_P \varphi^*(f) = \operatorname{ord}_Q(f)\, e_\varphi(P) = \operatorname{ord}_Q(\omega)\, e_\varphi(P)$

$$\underbrace{\deg \operatorname{div} \varphi^*(\omega)}_{2g_1 - 2} = \sum_{P \in C_1} \operatorname{ord}_P \varphi^*(\omega) = \sum_{Q \in C_2} \left(\sum_{P \in \varphi^{-1}(Q)} \left(\operatorname{ord}_Q(\omega)\, e_\varphi(P) + (e_\varphi(P) - 1) \right) \right)$$

$$= \sum_{Q \in C_2} \operatorname{ord}_Q(\omega) \deg(\varphi) + \sum_{P \in C_1} (e_\varphi(P) - 1)$$

$$= \deg(\varphi) \underbrace{\deg(\operatorname{div}(\omega))}_{2g_{C_2} - 2} + \sum_{P \in C_1} (e_\varphi(P) - 1)$$

$\boxtimes$

Formulation ABC-conjecture for curves

Rational ABC

$A+B=C \qquad (A,B,C)=1$

$u=\frac{A}{C} \qquad v=\frac{B}{C} \qquad u+v=1 \qquad u,v\in\mathbb{Q}$

log's palle van ABC-formulen.

$$\max(ht(u),ht(v)) \leq M_\varepsilon + (1+\varepsilon)\Big(\sum_{p\mid ABC}\log(p)\Big)$$

$$ht(u)=\max(\log|A|,\log|C|)$$

Curves

u,v non-constants in $k(C)$

$u+v=1$

$$\mathrm{div}(u)=A-C$$
$$\mathrm{div}(v)=B-C$$

$\deg(A)=\deg(B)=\deg(C)$
$=[\bar{k}(C):k(u)]$

geeft curve analogon van "hoogte"

$$\underbrace{k(u)\subset K}\subset k(C)$$

max. separable extension.

$\dim=\deg_s(u)=\deg_s(v)\leqslant[k(C):k(u)]$

ABC for curves

$$\boxed{\deg_s(u)=\deg_s(v)\leqslant(2g_C-2)+\sum_{P\in\mathrm{Supp}(A+B+C)}\deg(P)}$$

ABC for curves

$\boxed{\text{Case 1}}$: $k(u) — k(\mathcal{C})$ is separable

<u>Hurwitz</u>:

$$2g_{\mathcal{C}} - 2 \geqslant \deg_s(u)(-2) + \sum_{P \in \mathcal{C}} (e_\varphi(P) - 1)\deg P$$

$$\geqslant -2\deg_s(u) + \sum_{P \in \mathrm{Supp}(A+B+C)} (e_\varphi(P) - 1)\deg P$$

$$\geqslant -2\deg_s(u) + 3\deg_s(u) - \sum_{P \in \mathrm{Supp}(A+B+C)} \deg P$$

herschrijven geeft

$$\deg_s(u) \leqslant 2g_{\mathcal{C}} - 2 + \sum_{P \in \mathrm{Supp}(A+B+C)} \deg P \qquad \boxtimes$$

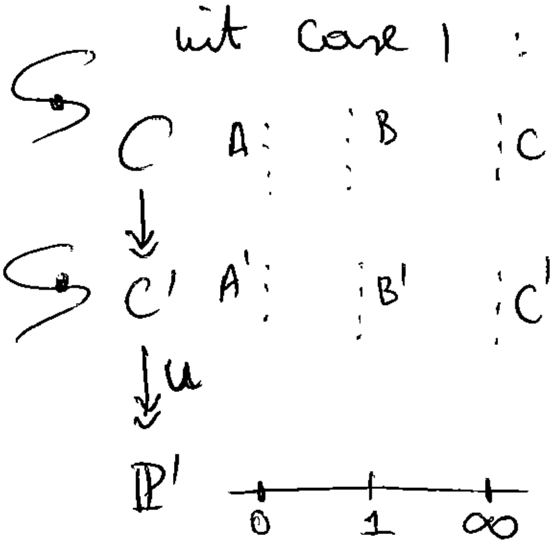
$\boxed{\text{Case 2}}$

$$k(u) \underbrace{\overline{\quad\quad\quad}}_{} K \quad\quad k(\mathcal{C})$$

$$k(u) \overbrace{———}^{\text{max sep}} K \overbrace{———}^{\text{zuiver insep.}} k(\mathcal{C})$$

$$K = k(\mathcal{C}')$$

uit case 1: $\deg_s(u) \leqslant 2g_{\mathcal{C}'} - 2 + \sum_{P' \in \mathrm{Supp}(A'+B'+C')} \deg P'$



<u>CLAIM</u>: $g_{\mathcal{C}'} = g_{\mathcal{C}}$

$P \leftrightarrow P'$ $\quad \deg P' = \deg P$

proof claim

$$K \subset M_1 \subset M_2 \subset \cdots \subset M_r = k(C)$$

$$M_{i+1}^p = M_i$$

thus $(\cdot)^p$ gives field iso $M_i \cong M_{i+1}$

genus is field inv $\Rightarrow g_{C'} = g_C$

pts are field inv $\Rightarrow \exists$ bij $P \longleftrightarrow P'$

$\deg(P) = \deg(P')$ so in each

$$M_i \subset M_{i+1}$$

$$P_i \longleftrightarrow P_{i+1}$$

$$t_{P_i} = t_{P_{i+1}}^p \quad \text{so} \quad e_{P_{i+1}} = p$$

$$e_{P_{i+1}} \cdot \deg(P_{i+1}) = p \cdot \deg(P_i)$$

$\boxtimes$

$\boxed{\mathbb{F}_1 \text{ en ABC von } \mathbb{Z}}$

$u+v=1$ $\qquad\qquad$ $q, q' \in \mathbb{Q}$

$$\begin{array}{ccc} C & & \mathrm{Spec}(\mathbb{Z}) \\ \downarrow u & & \downarrow q \\ \mathbb{P}^1_k & & \mathbb{P}^1/\mathbb{F}_1 \end{array}$$

- (1) $\mathbb{P}^1$ scheme pts

	$[0]$	$[1]$	$[2]$	$[n]$	$[\infty]$
deg	1	1	1	$\varphi(n)$	1

(2) $\mathrm{Spec}(\mathbb{Z})$

(2)	(3)	(p)	∞
$\deg(2)$	$\deg(3)$	$\deg(p)$	1

because $\mathbb{Z}_p \subset \mathbb{Q}$ are DVR's

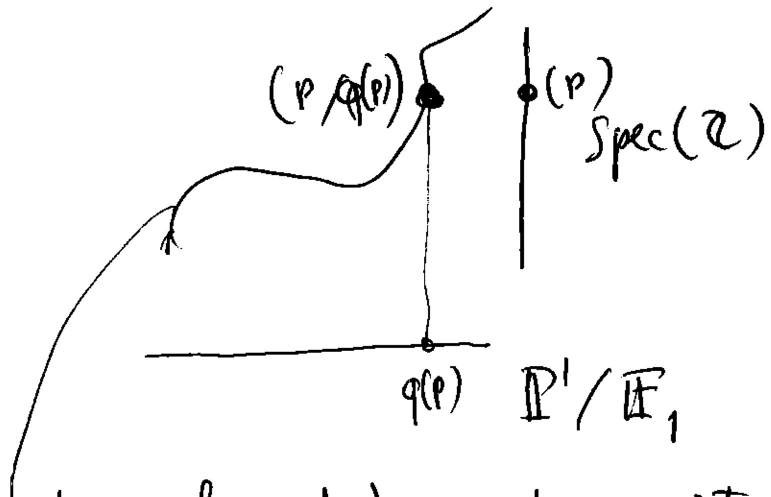
$-\log|q|$ addition real valuation

- $q \in \mathbb{Q}$ ~~$\nu_p(q)$~~ $\quad q = \dfrac{p_1^{e_1} \cdots p_2^{e_2}}{q_1^{f_1} \cdots q_s^{f_s}}$

$$\deg \mathrm{div}(q) = \sum e_i \deg(p_i) - \sum f_j \deg(q_j) - \log|q| \cdot 1$$

<u>Want</u> $\deg \mathrm{div}(q) = 0$ is satisfied if

$$\boxed{\deg(p) = \log(p)}$$



graph of q should be subvariety of
$\mathrm{Spec}(\mathbb{Z}) \times \mathbb{P}^1/\mathbb{F}_1$

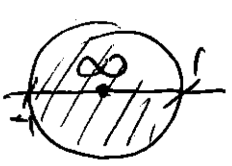
- ③ <u>definition of</u> $q : \mathrm{Spec}(\mathbb{Z}) \longrightarrow \mathbb{P}^1/\mathbb{F}_1$

$$q = \frac{p_1^{e_1} \cdots p_n^{e_n}}{q_1^{f_1} \cdots q_l^{f_l}} = \frac{a}{b}$$

$$p_i \longrightarrow [0]$$
$$q_j \longrightarrow [\infty]$$

- $p \notin \{p_i, q_j\}$ dan $\bar{a}, \bar{b} \in \mathbb{F}_p^*$

$$p \longrightarrow \text{orde}\left(\frac{\bar{a}}{\bar{b}}\right) \text{ in } \mathbb{F}_p^*$$



$\infty \longrightarrow 0$ if $a<b$ $\quad \infty$ if $a>b$

± 1 cont flts <u>motivatie</u> $[n] \longleftrightarrow \mathbb{Q}(\varepsilon)$ met ε prim n-th root

als $\text{orde}\left(\frac{\bar{a}}{\bar{b}}\right) = n$ in $\mathbb{F}_p^*$ $\Rightarrow \exists$ priemideaal
$P \lhd \mathbb{Q}(\varepsilon)$ met $P \cap \mathbb{Z} = (p)$

and $\frac{a}{b} - \varepsilon \in P\mathcal{O}_P$ dan $\frac{a}{b}(P) = \varepsilon(P)$.

a)b: is $q = \frac{a}{b}$: $\mathrm{Spec}(\mathbb{Z}) \longrightarrow \mathbb{P}^1/\mathbb{F}_1$ cover ?e

wat is $\deg(q)$

analogy curve case $\deg(q) = \deg(a) = \log |a|$

$$q(p) = n \quad \text{iff} \quad \left(\frac{\bar{a}}{\bar{b}}\right)^n = 1 \text{ in } \mathbb{F}_p^*$$

$\Leftrightarrow p \mid a^n - b^n$ én $p \nmid a^m - b^m$ als $m < n$

(of $m \mid n$)

dus: boven iedere n : eindig # p's

$\boxed{\text{Zsigmondy Thm}}$

$(a,b) = 1 \qquad 1 \leq b < a$

$\Rightarrow \forall n > 1 \ \exists p \mid a^n - b^n$ and $p \nmid a^m - b^m$

for $m < n$

behalve

$$\begin{cases} ① \ a = 2, \ b = 1, \ n = 6 \\ ② \ a + b = 2^k \text{ en } n = 2 \end{cases}$$

dus "meestal" ontstaat een cover!

herschrijven Hurwitz van $C \xrightarrow{f} \mathbb{P}^1_k$

$$2g_C - 2 \geq -2\deg(f) + \sum^{\text{Schema}} (e_f(P) - 1)\deg(P)$$

$$\boxed{\sum^{\text{schema}}_{\text{defect}} \underbrace{\frac{(e_f(P)-1)\deg(P)}{\deg(f)}}_{\delta_P} \leq 2 - \frac{2-2g_C}{\deg(f)}}$$

wederom: mogen $\sum$ over maximale ptn nemen.

$$\mathrm{Spec}(\mathbb{Z}) \xrightarrow{q} \mathbb{P}^1/\mathbb{F}_1$$

$\boxed{\text{wat is } e_q(p)\,?}$

$$p \in q^{-1}([0]) \Rightarrow \quad v_p(a) \quad = e_q(p)$$

$$p \in q^{-1}([\infty]) \Rightarrow \begin{cases} + v_q(b) & = e_q(p) \\ \log(q) & = e_q(\infty) \end{cases}$$

$$p \in q^{-1}([n]) \Rightarrow \quad e_q(p) = k \text{ if } \quad p^k \mid a^n - b^n$$

en p^{k+1} k maximaal.

$$\boxed{\text{arithmetic defect} \quad \delta(p) = \frac{(e_q(p)-1)\log(p)}{\log(a)}}$$

$$\delta([d]) = \sum_{p \in q^{-1}([d])} \delta(p)$$

$$\begin{cases} a = p_1^{e_1} \cdots p_k^{e_k} & b = q_1^{f_1} \cdots q_s^{f_s} \\ a_0 = p_1 \cdots p_k & b_0 = q_1 \cdots q_s \\ a_1 = p_1^{e_1 - 1} \cdots p_k^{e_k - 1} & b_1 = q_1^{f_1 - 1} \cdots q_s^{f_s - 1} \end{cases}$$

- $\delta_{[0]} = \dfrac{\log(a_1)}{\log(a)}$
- $\delta_{[\infty]} = \dfrac{\log(b_1) + \overbrace{\log(q) - 1}^{\text{bij } \log \infty}}{\log(a)}$

- $\delta_{[1]} = \dfrac{\cancel{\text{to}} \log(a-b)_1}{\log(a)}$

$$q^{-1}[1] = \{ p \mid a-b \}$$

<u>Stel:</u> Hurwitz geldt voor $\mathrm{Spec}(\mathbb{Z}) \xrightarrow{q} \mathbb{P}^1$ dan

$$\delta_{[0]} + \delta_{[\infty]} + \delta_{[1]} \leqslant 2 - \frac{2 - 2g_{\mathrm{Spec}(\mathbb{Z})}}{\log(a)}$$

dus $\forall \varepsilon$, $\exists\, C(\varepsilon)$: $\forall\, 1 \leq b < a$ $\quad (a,b) = 1$

$$\frac{1}{\log(a)} \Big[\log(a_1) + \log((a-b)_1) + \log(b_1) + \log(a) - \log(b) - 1 \Big]$$

$$\leqslant 2 + \varepsilon + \frac{C(\varepsilon)}{\log(a)}$$

Neem nu: $A+B=C$ $\quad (A,B,C)=1$

$a = C$ en $b = \min(A,B)$

en neem $q = \frac{a}{b} : \mathrm{Spec}(\mathbb{Q}) \to \mathbb{P}^1/\mathbb{F}_1$

voor deze a e b hebben we $a - b \geq \frac{a}{2}$

$$C - \underbrace{\min(A,B)}_{\leq \frac{C}{2}} \geq \frac{C}{2}$$

$$\log a_1 = \log a - \log a_0$$

$$\odot \quad \frac{\log a_1}{\log a} = \underbrace{1 - \frac{\log a_0}{\log a}}_{*}$$

$$\log b_1 = \log b - \log b_0$$

$$\odot \quad \frac{\log b_1 + \log a - \log b - 1}{\log a} = \underbrace{1 - \frac{\log b_0}{\log a} - \frac{1}{\log a}}_{**}$$

$$\odot \quad \frac{\log((a-b)_1)}{\log a} = \frac{\log(a-b) - \log((a-b)_0)}{\log a}$$

$$a - b \geq \frac{a}{2} \quad \text{dus} \quad \log(a-b) \geq \log(a) - \log(2)$$

$$\geq \underbrace{1 - \frac{\log((a-b)_0)}{\log(a)} - \frac{\log(2)}{\log(a)}}_{***}$$

optellen levert

$$* + ** + *** \leq 2 + \varepsilon + \frac{C(\varepsilon)}{\log(a)}$$

$$3 - \frac{\log(a_0 b_0 (a-b)_0)}{\log(a)} \leq 2 + \varepsilon + \frac{C(\varepsilon) + \log(2) + 1}{\log(a)}$$

$$= 2 + \varepsilon + \frac{\log C'(\varepsilon)}{\log(a)}$$

12 bis

$$\delta_{[0]} + \delta_{[1]} + \delta_{[\infty]} \leq 2 + \frac{c}{\log(a)}$$

$$C = 2g_{Spec} - 2$$

$$1 - \frac{\log(a_0 b_0 (a-b)_0)}{\log(a)} \leq 0 + \frac{C + \log(2) + 1}{\log(a)}$$

$$1 \leq \frac{\log a_0 b_0 (a-b)_0}{\log(a)} + \frac{\log C'}{\log(a)}$$

$$\log(a) \leq \log C'(a_0 b_0 (a-b)_0)$$

$$a \leq C'(a_0 b_0 (a-b)_0)$$

$$c \leq C' (\mathrm{rad}\, ABC)$$

$$1-\varepsilon \leq \frac{\log(a_0 b_0 (a-b)_0)}{\log(a)} + \frac{\log c'(\varepsilon)}{\log a}$$

$$(1-\varepsilon)\log a \leq \log\left(C'(\varepsilon)\, a_0 b_0 (a-b)_0\right)$$

$$a^{1-\varepsilon} \leq C'(\varepsilon)\, a_0\, b_0 (a-b)_0$$

$$\cancel{a^{\varepsilon+\varepsilon^2} \leq \left(C'(\varepsilon)\, a_0 b_0 (a-b)_0\right)^{\varepsilon}}$$

$$\cancel{a^{1-\varepsilon^2} \leq C'(\varepsilon)^{1+\varepsilon} \left(a_0 b_0 (a-b)_0\right)}^{1+\varepsilon}$$

$$\cancel{a^{\varepsilon} a^{(1-\varepsilon)}} \leq a^{\varepsilon}$$

$$\cancel{a^{\varepsilon} \leq a^{1-\varepsilon}}$$

$$a^{1-\varepsilon^2} \leq$$