$-\otimes_{\mathbb{F}_1} \mathbb{Z}$ = stripping structure

$$\begin{tikzcd} \mathbb{F}_1\text{-alg} = \text{Flrings} \arrow[r, bend left, "-\otimes_{\mathbb{F}_1}\mathbb{Z}"] & \mathbb{Z}\text{-alg} = \text{Rings} \arrow[l, bend left, "?"] \end{tikzcd}$$

? must be right adjoint of $-\otimes_{\mathbb{F}_1} \mathbb{Z}$

$U:$ Flrings $\longrightarrow$ Rings

forgetfull functor

$\boxed{\text{want}}$: for every Flring A and Ring B

$$\mathrm{Hom}_{\text{Flring}}(A, ?(B)) = \mathrm{Hom}_{\text{Ring}}(U(A), B)$$

Thm $?(B) = \Lambda(B)$ the ring of Big Witt vectors of B.

$\boxed{\text{Construction of } \Lambda(A) \quad \text{for } A \text{ ring}}$

$$\Lambda(A) = 1 + t\,A[[t]]$$

i.e. formal power series / A with constant coefficient 1.

$\boxed{\text{addition}}$ = usual multiplication of formal power series

$$u(t) \oplus v(t) = u(t) \times v(t)$$

$$\text{"}0\text{"} = 1$$

$$\text{"}\ominus\text{"}\, u(t) = u(t)^{-1}$$

$\boxed{\text{multiplication}}$ = induced by the rule that for all $a, b \in A$

$$\frac{1}{(1-at)} \otimes \frac{1}{(1-bt)} = \frac{1}{1-abt}$$

$$\text{"}1\text{"} = \frac{1}{1-t} = 1 + t + t^2 + \dots$$

for any power series

(*) $1+a_1t+a_2t^2+\dots$

exists $\alpha_1, \alpha_2, \dots \in A$ s.t.

$$(*) = \prod_{i=1}^{\infty} \frac{1}{(1-\alpha_i t^i)}$$

$$(1+\alpha_1 t+\alpha_1^2 t^2+\dots)(1+\alpha_2 t^2+\alpha_2^2 t^4+\dots)(1+\alpha_3 t^3+\dots$$

$$= 1+\alpha_1 t+(\alpha_1^2+\alpha_2)t^2+(\alpha_1^3+\alpha_3)t^3+\dots$$

$\alpha_1 = a_1$

$\alpha_2 = a_2 - a_1^2$

$\alpha_3 = a_3 - a_1^3$

etc.

for every n have in $A[\omega][\sqrt[n]{\alpha_n}]$ $\omega^n=1$

$$1-\alpha_n t^n = \prod_{i=1}^{n}(1-\sqrt[n]{\alpha_n}\,\omega^i t)$$

So in

$$A[M][\sqrt[1]{\alpha_1}, \sqrt[2]{\alpha_2}, \sqrt[3]{\alpha_3}, \dots]$$

can write "formally"

$$(*) = \frac{1}{1-\alpha_1 t}\cdot\left(\frac{1}{1-\sqrt[2]{\alpha_2}t}\cdot\frac{1}{1+\sqrt[2]{\alpha_2}t}\right)\cdot\left(\frac{1}{(1-\sqrt[3]{\alpha_3}\,t)(1-\sqrt[3]{\alpha_3}\,\xi\,t)(1-\sqrt[3]{\alpha_3}\,\xi^2 t)}\right)\cdot$$

such that for all N

$$* \bmod t^{N+1} = 1 + a_1 t + a_2 t + \cdots + a_N t^N$$

$$= \left(\frac{1}{1-\alpha_1 t}\right)\left(\frac{1}{(1-\sqrt[2]{\alpha_2}\, t)(1+\sqrt[2]{\alpha_2}\, t)}\right)\cdots\left(\frac{1}{(1-\sqrt[n]{\alpha_n}\, t)(\cdots\cdots(1-\sqrt[n]{\alpha_n}\,\omega^{n-1} t)}\right)$$

$$\bmod t^{N+1}$$

But then can compute

$$(1+a_1 t + a_2 t^2 + \cdots) \otimes (1 + b_1 t + b_2 t^2 + \cdots)$$

by working out modulo each t^{N+1}

$$\left[\underbrace{\left(\frac{1}{1-\alpha_1 t}\right)}_{A_1} \oplus \underbrace{\left(\frac{1}{1-\sqrt[2]{\alpha_2}\, t}\right) \oplus \left(\frac{1}{1+\sqrt[2]{\alpha_2}\, t}\right)}_{A_2} \oplus \cdots \oplus \underbrace{\left(\frac{1}{1-\sqrt[N]{\alpha_N}\, t}\right) \oplus \cdots \oplus \left(\frac{1}{1-\sqrt[N]{\alpha_N}\,\omega^{N-1} t}\right)}_{A_N}\right]$$

$$\otimes \left[\underbrace{\qquad}_{B_1}\ \underbrace{\qquad}_{B_2}\ \cdots\ \underbrace{\qquad\qquad}_{B_N}\right]$$

and terms in A_m, B_m with $m > N$ will only involve terms in $A \otimes B$ of degree $> N$.

$\Lambda(A)$ is Fling with Adams operations induced by

$$\psi^n\left(\frac{1}{1-at}\right) = \frac{1}{1-a^n t}$$

Dus voor algemene machreeks

$$x = 1+a_1t+a_2t^2+\dots = A_1 \oplus A_2 \oplus \dots$$

$$\psi^n(x) = \psi^n(A_1) \oplus \psi^n(A_2) \oplus \dots$$

dus make dat ψ^n additief is

is ook multiplicatief

$$\psi^n\left(\frac{1}{1-at} \otimes \frac{1}{1-bt}\right)$$

$$= \psi^n\left(\frac{1}{1-abt}\right) = \frac{1}{1-a^nb^nt}$$

$$= \frac{1}{1-a^nt} \otimes \frac{1}{1-b^nt}.$$

Is Frobenius lift ?

$$\psi^n\left(\frac{1}{1-at}\right) = \frac{1}{1-a^n t}$$

$$\left(\frac{1}{1-at}\right)^{\otimes n} = \frac{1}{1-a^n t}$$

in algemeen

$$\psi^p\left(\frac{1}{1-a_1 t} \oplus \cdots \oplus \frac{1}{1-a_i t}\right) = \frac{1}{1-a_1^p t} \oplus \cdots \oplus \frac{1}{1-a_i^p t}$$

$$\left(\frac{1}{1-a_1 t} \oplus \cdots \oplus \frac{1}{1-a_i t}\right)^{\otimes p}$$

is p-voud uit binomiaal formule.

du

Define maps

$$\gamma_n : \Lambda(A) \longrightarrow A$$

via $$\frac{t\,u'}{u} = \sum_{n=1}^{\infty} \gamma_n(u)\, t^n$$

$$u = 1 + a_1 t + a_2 t^2 + \cdots$$

$$t u' = a_1 t + 2 a_2 t^2 + 3 a_3 t^3 + \cdots$$

$\cancel{a^{-1} = 1}$ $\quad u^{-1} = 1 - a_1 t + \cdots$

$$(1 + a_1 t + a_2 t^2 + \cdots)(1 + a_1 t + \cdots$$

$$\frac{t u'}{u} = (1 - a_1 t + \cdots)(a_1 t + 2 a_2 t^2 + \cdots$$

$$=$$

$\boxed{\text{logarithmic derivative}}$ $\frac{u'}{u}$

transforms multiplication in addition

$$\frac{(uv)'}{uv} = \frac{uv' + u'v}{uv} = \frac{v'}{v} + \frac{u'}{u}$$

for $u = \dfrac{1}{1-at}$

$u' = \dfrac{+a}{(1-at)^2}$ $\Bigg\} \Rightarrow \dfrac{u'}{u} = \dfrac{a}{1-at}$

thus $\dfrac{tu'}{u} = ta(1 + at + a^2t^2 + \ldots$

$= ta + a^2t^2 + a^3t^3 + \ldots$

$\Rightarrow \gamma_n\left(\dfrac{1}{1-at}\right) = a^n$ is multiplicative in a

$\Rightarrow \gamma_n\left(\dfrac{1}{1-at} \otimes \dfrac{1}{1-bt}\right)$

$= \gamma_n(\quad) \cdot \gamma_n(\quad) = a^n \cdot b^n$

$\Rightarrow$ ~~bys~~ before

$\forall n$: $\gamma_n : \Lambda(A) \longrightarrow A$ is a morphism.

If A is Fling
$\exists$ Fling-morfisme σ_t making diagram com.

$$
\begin{tikzcd}
 & \Lambda(A) = 1 + tA[[t]] \arrow[d, "(\gamma_1, \gamma_2, \gamma_3, \ldots)"] \\
A \arrow[ur, bend left, "\sigma_t"] \arrow[r, "\Psi = (\psi^1, \psi^2, \psi^3, \ldots)"'] & A^{\omega} = (A, A, A, \ldots)
\end{tikzcd}
$$

That is

$$\sigma_t(a) = \exp\left(\int \frac{1}{t} \sum_{n=1}^{\infty} \psi^n(a) t^n\right)$$

$$= \exp\left(\int \sum_{n=0}^{\infty} \psi^n(a) t^{n-1}\right)$$

$$= \exp\left(\sum_{n=0}^{\infty} \frac{1}{n} \psi^n(a) t^n\right)$$

easy to check $\sigma_t(a+b) = \sigma_t(a) \oplus \sigma_t(b)$

more difficult $\sigma_t(a \cdot b) = \sigma_t(a) \otimes \sigma_t(b)$

en compatible met ψ^n-maps

adjointness follows

A Fling $\qquad$ B Ring

$$\underbrace{\mathrm{Hom}_{\mathbb{F}_1}(A, \Lambda(B))}_{} = \underbrace{\mathrm{Hom}_{\mathbb{Z}}(A \otimes_{\mathbb{F}_1} \mathbb{Z}, B)}_{}$$

algebra maps $A \to \Lambda(B)$ respecting Fl-str. $\qquad$ algebra map $A \to B$ forgetting Fl-structure of A

$A \xrightarrow{f} B$ algebra map

$\Downarrow$

$\Lambda(A) \xrightarrow{\tilde{f}} \Lambda(B)$ algebra map resp. Fl-structure

$1 + \sum a_i t^i \mapsto 1 + \sum f(a_i) t^i$

$\Downarrow$

$A \xrightarrow{\sigma_t} \Lambda(A) \xrightarrow{\tilde{f}} \Lambda(B)$ is algebra m

$A \to \Lambda(B)$ respecting Fl

$1 + \sum b_i t^i \mapsto b_1 = \gamma_1$

Conversely $A \xrightarrow{g} \Lambda(B) \xrightarrow{\gamma_1} B$ gives alg. map

Fl-map

$$
\begin{tikzcd}
W(A) \arrow[r, "\gamma"] \arrow[d, equal] & \Lambda(A) \arrow[d, bend left, "\frac{u'}{u}"] \\
A^{\infty} \arrow[r, "=", "\text{ident.}"'] & tA[[t]]
\end{tikzcd}
$$

$$(a_1, a_2, \dots) \mapsto \prod \frac{1}{1 - a_i t^i}$$

comp. multi addi

Hadamard product.

$$w \oplus w' = \gamma^{-1}(\gamma(w) + \gamma(w'))$$

$$w \odot w' = \gamma^{-1}(\gamma(w) \cdot \gamma(w'))$$

$W(A)$ is ring of big Witt vectors.

$W(\mathbb{Z})$ als Burnside Ring

$\exp^{\mathbb{Z}} = C = \langle 1, t, t^2, \ldots \rangle$ ∞ cyclic grp written multiplicatively

X C-set is called cyclic set

$C(n)$ cycle of length n $n \in \mathbb{N} \cup \{\infty\}$

- C-set X without infinite cycles and such that for each n only fin. many orbits $C(n)$ is called "almost finite"

MAJOR EXAMPLE

$X(\overline{\mathbb{F}}_p)$ X varietät

C acts via Frob. auto on coordinates

- $\Rightarrow X(\overline{\mathbb{F}}_p)$ is almost finite C-set.

X_1, X_2 almost finite

$+$ $\Rightarrow X_1 \sqcup X_2$
$\times$ $X_1 \times X_2$ $\Big\}$ almost finite

So have ring structure on isoclasses of almost fin. sets $\widehat{\Omega}(C)$ Burnside ring

$[X] \in \widehat{\Omega}(C)$ represents X almost fin C-set from now on X almost finite C-set (2

$$\varphi_{C^n}(X) = \#\{x \in X : t^n \cdot x = x\}$$

(= # elements lying in orbits of sizes divisors of n, so finite)

gives ring morphism ($\sqcup \to +$ $\times \to \cdot$)

$$\varphi_{C^n} : \hat{\Omega}(C) \longrightarrow \mathbb{Z}$$

Have collective ring morphism

$$\hat{\varphi} = \prod_{n \in \mathbb{N}} \varphi_{C^n} : \hat{\Omega}(C) \longrightarrow \underbrace{\mathbb{Z}^{\mathbb{N}}}_{\substack{\text{all maps } \mathbb{N} \to \mathbb{Z} \\ \text{with comp. addition} \\ \text{and multiplication}}} = \underbrace{gh(C)}_{\text{ghost ring}}$$

$\hat{\varphi}$ injective but NOT surjective

$d = d(i) \in \operatorname{Im} \hat{\varphi}$ iff for all $n \in \mathbb{N}$

$$\sum_{i=1}^{n} d(\gcd(i,n)) = \sum_{i|n} \varphi\left(\frac{n}{i}\right) \cdot d(i) \equiv 0 \mod n$$

Have commutative diagram

$$\begin{tikzcd}
 & Nr(\mathbb{Z}) \text{ ``necklace algebra''} \arrow[d, "\text{interpretation}"] & \\
W(\mathbb{Z}) \arrow[r, "\tau"] \arrow[d, "\Phi"'] & \hat{\Omega}(C) \arrow[r, "s_t"] \arrow[d, "\hat{\varphi}"'] & \Lambda(\mathbb{Z}) \arrow[d, "L_z"] \\
\prod_{\mathbb{N}} \mathbb{Z} \arrow[r, "\text{obvious}"] & gh(C) \arrow[r, "\text{identif.}"] & t\,\mathbb{Z}[[t]]
\end{tikzcd}$$

Here L_z is logarithmic derivative

$$a(t) \longrightarrow L_z(a(t)) = t \cdot \frac{d}{dt} \log a(t) = t\, \frac{a'(t)}{a(t)}$$

τ and s_t are combinatorially defined.

$Nr(\mathbb{Z})$ necklace algebra is set $\mathbb{Z}^{\mathbb{N}}$

with <u>addition</u> : componentwise

<u>multiplication</u> :

$$b = (b_1, b_2, \ldots$$
$$b' = (b'_1, b'_2, \ldots$$

$$(b \cdot b')_n = \sum_{\mathrm{lcm}(i,j)=n} (i,j)\, b_i b'_j$$

$(\, , \,) = \gcd(\, , \,)$

$[\, , \,] = \mathrm{lcm}(\, , \,)$

<u>interpretation map</u>

$$Nr(\mathbb{Z}) = \mathbb{Z}^{\mathbb{N}} \longrightarrow \hat{\Omega}(C)$$

$$\underline{b} = (b_1, b_2, \dots) \longmapsto X(\underline{b}) = \sum_{n=1}^{\infty} b_n \cdot C_n \in \hat{\Omega}(C)$$

$$= X_+(b) = \bigsqcup_{n \in \mathbb{N}_+} b_n \cdot C_n$$

$$- X_-(b) = \bigsqcup_{n \in \mathbb{N}_-} (-b_n) \cdot C_n$$

- is a ring-morphism.

<u>Composition</u> ghost-map

$$Nr(\mathbb{Z}) = \mathbb{Z}^{N} \xrightarrow{\ itp\ } \hat{\Omega}(C) \xrightarrow{\ \hat{\varphi}\ } gh(C) = \mathbb{Z}^{\mathbb{N}}$$

$$gh \quad \underline{b} \longmapsto \underline{d}$$

- $$d_n = \varphi_{C^n}(X(b)) = \varphi_{C^n}(X_+(b)) - \varphi_{C^n}(X_-(b))$$

sequence $\underline{d}$ is related to $\underline{b}$ via

$$\prod_{n=1}^{\infty} \left(\frac{1}{1-t^n} \right)^{b_n} = \exp\left(\int \sum_{n=1}^{\infty} d_n t^{n-1} \, dt \right)$$

$$gh(b) = \hat{b} \quad \text{where} \quad (\hat{b})_n = \sum_{i|n} i \cdot b_i$$

X almost finite cyclic set

$$S^n(X) = \{ g : X \to \mathbb{N}_0 = \mathbb{N} \cup \{0\} \}$$

with finite support and s.t.

$$\sum_{x \in X} g(x) = n$$

is called "symmetric power" of X

$S^n(X)$ is again almost finite and

$$S^n(X_1 \cup X_2) = \bigsqcup_{i+j=n} S^i(X_1) \times S^j(X_2)$$

This relation implies that

<u>Map</u> $s_t : \hat{\Omega}(C) \longrightarrow \Lambda(\mathbb{Z}) = 1 + t\mathbb{Z}[[t]]$

$$X \mapsto s_t(X) = 1 + \varphi_C(S^1X)t + \varphi_C(S^2X)t^2 + \dots$$

$$\varphi_C = \varphi_{C'} = \#\text{fixpts}$$

$$s_t(X_1 \cup X_2) = s_t(X_1) \cdot s_t(X_2)$$

is isomorphism and if

$$s_t(X(\underline{b})) = 1 + \sum_{n=1}^{\infty} a_n t^n$$

then $\underline{b}$ and $\underline{a}$ are related

$$\prod_{n=1}^{\infty} \left(\frac{1}{1-t^n}\right)^{b_n} = 1 + \sum_{n=1}^{\infty} a_n t^n .$$

~~AAA-pitt~~

"congruence maps"

$q_1 \in \mathbb{N}$ $\quad q_1^{(C)} = \{ g : C \to \{1, .., q_1\} \}$
s.t. $\exists n \in \mathbb{N}$ and if
$z_1 z_2^{-1} \in C^n = \{ t^n \mid t \in C \}$
$\Rightarrow g(z_1) = g(z_2)$

- $q_1^{(C)}$ is almost finite set with action

$$t.g : \quad C \to \{1, .., q_1\}$$
$$z \mapsto g(t^{-1} z)$$

and check that

- $$\varphi_{C^n}(q_1^{(C)}) = q_1^n$$
$$\| \qquad\qquad \|$$
$$\# \{1, .., q_1\}^{C/C^n} = \# \{1, .., q_1\}^{C(n)}$$

gives map $\quad \mathbb{N} \to \hat{\Omega}(C)$

extends to $\quad \mathbb{Z} \to \hat{\Omega}(C)$
$$q_1 \to q_1^{(C)}$$

en $\hat{\varphi}(q_1^{(C)}) = (q_1, q_1^2, q_1^3, \ldots)$

en dus $s_t(q_1^{(C)}) = \exp\left(\int q_1^n t^{n-1} dt\right)$

$$= \frac{1}{1-q_1 t} = 1 + q_1 t + q_1^2 t^2 + \dots$$

extend $\mathbb{Z} \longrightarrow \hat{\Omega}(C)$

to map $W(\mathbb{Z}) \longrightarrow \hat{\Omega}(C)$

- X almost finite set, $n \in \mathbb{N}$

C
$\cup$
C^n

induction wrt n-th power map $\sigma_n : C \to C$, $t \mapsto t^n$

consider

$\mathrm{ind}_n X$ ~~the~~ $C \times X$

with C-structure

- $t.(c,x) = (ct^{-n}, tx)$

$\underbrace{\mathrm{ind}_n X}$ = set of C-orbits in $C \times X$ for this action

becomes again C-set via action

$t.\mathcal{O}(c,x) = \mathcal{O}(tc, x)$

$$\begin{cases} \mathrm{ind}_n(X_1 \cup X_2) \cong \mathrm{ind}_n X_1 \cup \mathrm{ind}_n X_2 \\ \mathrm{ind}_n(C(i)) \cong C(ni) \end{cases}$$

have $\varphi_{C^m}(\mathrm{ind}_n X) = \begin{cases} n \cdot \varphi_{C^{m/n}}(X) & \text{if } n \mid m \\ 0 & \text{otherwise} \end{cases}$

- $$\tau : W(\mathbb{Z}) \longrightarrow \hat{\Omega}(C)$$

  $$\underline{q} = (q_1, q_2, \ldots) \mapsto \sum_{n=1}^{\infty} \mathrm{ind}_n (q_n^{(C)})$$

  with $\varphi_{C^m}(\tau(q)) = \sum_{d \mid m} d \cdot q_d^{m/d}$

  is ring-isomorphism

- If $\tau(\underline{q}) = X(\underline{b})$ the sequences are related

  $$\prod_{n=1}^{\infty} \frac{1}{1 - q_n t^n} = \prod_{n=1}^{\infty} \left(\frac{1}{1-t^n} \right)^{b_n}$$

Have also <u>restriction</u> $\quad C^n \subset C$

$\qquad\qquad\qquad\qquad\qquad\qquad \mathrm{res}_n X \quad X$

X almost finite set

$X = \mathrm{res}_n X$ is alm finite set with action as set

$$t.x = t^n x$$

$$\begin{cases} \mathrm{res}_n(X_1 \cup X_2) = \mathrm{res}_n X_1 \sqcup \mathrm{res}_n X_2 \\ \mathrm{res}_n(X_1 \times X_2) = \mathrm{res}_n X_1 \times \mathrm{res}_n X_2 \end{cases}$$

Also res_n gives endo-morphism.

$$\hat{\Omega}(C) \to \hat{\Omega}(C)$$

These are the Adams operations on $\hat{\Omega}(C)$.

① $\mathrm{res}_n C(m) \cong (n,m).\, C\left(\frac{[n,m]}{n}\right)$

② $\mathrm{ind}_n C(m) = C(nm)$

③ $\mathrm{Ker}(\mathrm{res}_n) = \{ x \in \hat{\Omega}(C) \mid \varphi_{C^m}(x) = 0 \ \forall\ n \mid m \}$

④ $\mathrm{Im}(\mathrm{ind}_n) = \{ x \in \hat{\Omega}(C) \mid \varphi_{C^m}(x) = 0 \ \forall\ n \nmid m \}$

Have operators on $N_2(\mathbb{Z})$

Frobenius operator

$$f_n : N_2(\mathbb{Z}) \longrightarrow N_2(\mathbb{Z})$$

$$(b_1, b_2, \dots) \mapsto \Big(\sum_{[n,i]=n} (n,i) b_i , \sum_{[n,i]=2n} (n,i) b_i , \dots$$

- Verschiebung operator

$$v_n : N_2(\mathbb{Z}) \longrightarrow N_2(\mathbb{Z})$$

$$(b_1, b_2, \dots) \mapsto (\underbrace{0, \dots, 0}_{n-1}, b_1, \underbrace{0, \dots, 0}_{n-1}, b_2, 0 \dots$$

These operators commute with ind and res

-

$$\begin{tikzcd} N_2(\mathbb{Z}) \arrow[r, "it_p"] \arrow[d, "f_n"'] & \hat{\Omega}(C) \arrow[d, "res_n"] \\ N_2(\mathbb{Z}) \arrow[r, "it_p"] & \hat{\Omega}(C) \end{tikzcd}$$

$$\begin{tikzcd} N_2(\mathbb{Z}) \arrow[r, "it_p"] \arrow[d, "v_n"'] & \hat{\Omega}(C) \arrow[d, "ind_n"] \\ N_2(\mathbb{Z}) \arrow[r, "it_p"] & \hat{\Omega}(C) \end{tikzcd}$$