

Stripping off structure

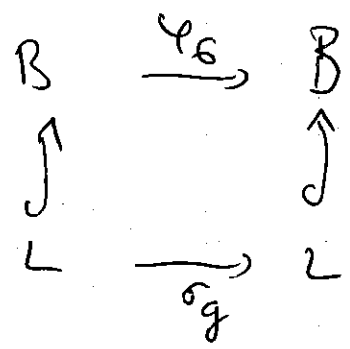
Aliens and Reality Question (1)

$k \subset L$ finite Galois with group G

want to study k -algebras by only considering L -algebras + extra structure.

B L -algebra is said to have G -structure iff

$\forall g \in G$ \exists automorphism $\varphi_g : B \rightarrow B$
extending Galois action by g on L



B, B' L -algebras with G -structure

$B \xrightarrow{\varphi} B'$ G -morphism φ

φ L -algebra morphism preserving G -struct

$$\varphi(g \cdot b) = g \cdot \varphi(b)$$

CLAIM:

L -algebras with G -structure and G -morphisms

\updownarrow equivalent

k -algebras

If we only consider L -algebra the functor

$$k\text{-alg} \rightarrow L\text{-alg} \quad A \rightarrow A \otimes_h L$$

can be viewed as "stripping" off G -structure.

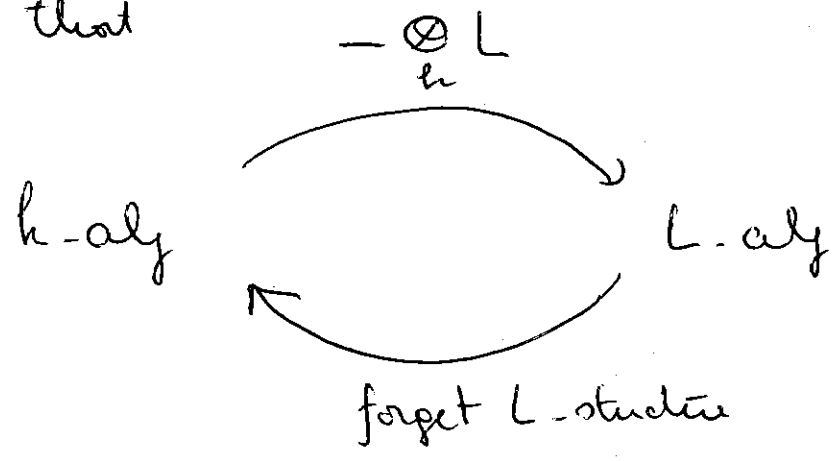
If A is k -algebra
 B is L -algebra

Have natural iso

$$\text{Alg}_k(A, B) = \text{Alg}_L(A \otimes_h L, B)$$

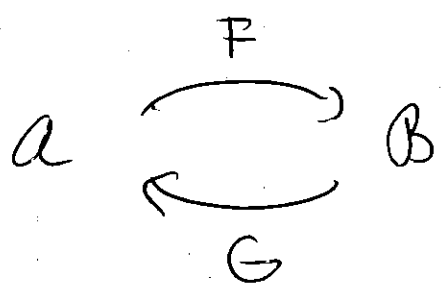
$$f \mapsto (a \otimes l \mapsto f(a)l)$$

This says that



are adjoint functors : forget L -structure is right adjoint of $- \otimes_h L$ and $- \otimes_h L$ is left adjoint of forgetting L -structure.

In general



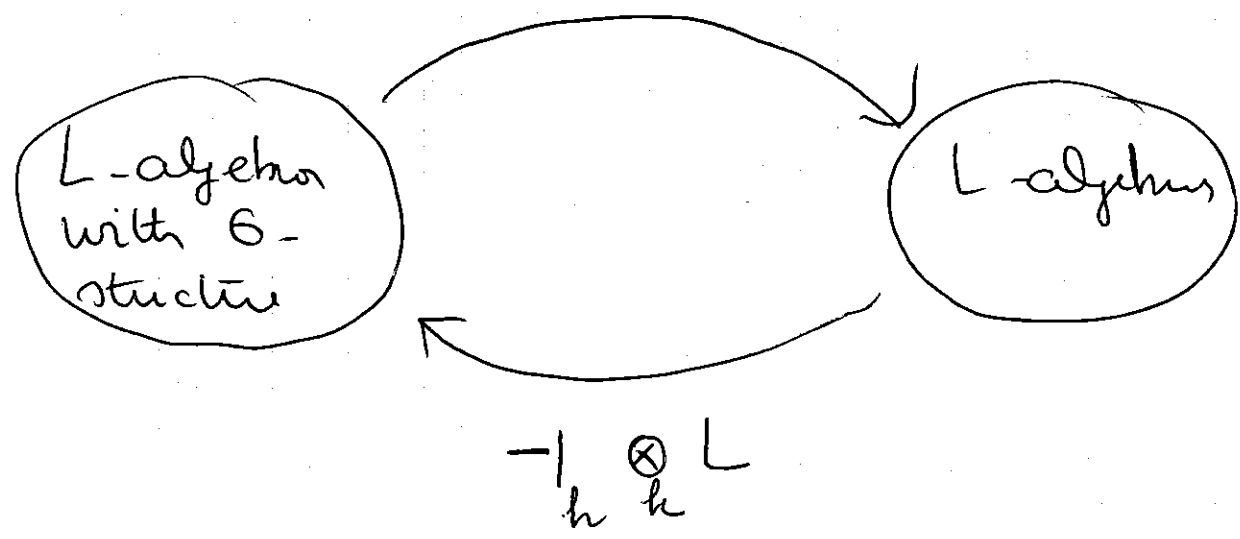
(F, G) is adjoint pair of functors iff

$$\forall A \in \text{Ob}(\mathcal{A}) \quad \forall B \in \text{Ob}(\mathcal{B})$$

$$\text{Hom}_{\mathcal{A}}(A, G(B)) = \text{Hom}_{\mathcal{B}}(F(A), B)$$

If only want to work with L-algebras

Forget G-struct



Allow to define geometry "under L" by only working with L-objects + extra structure.

Need to find right adjoint of "forget extra structure" to know what L-object is considered as "the object"

Flings

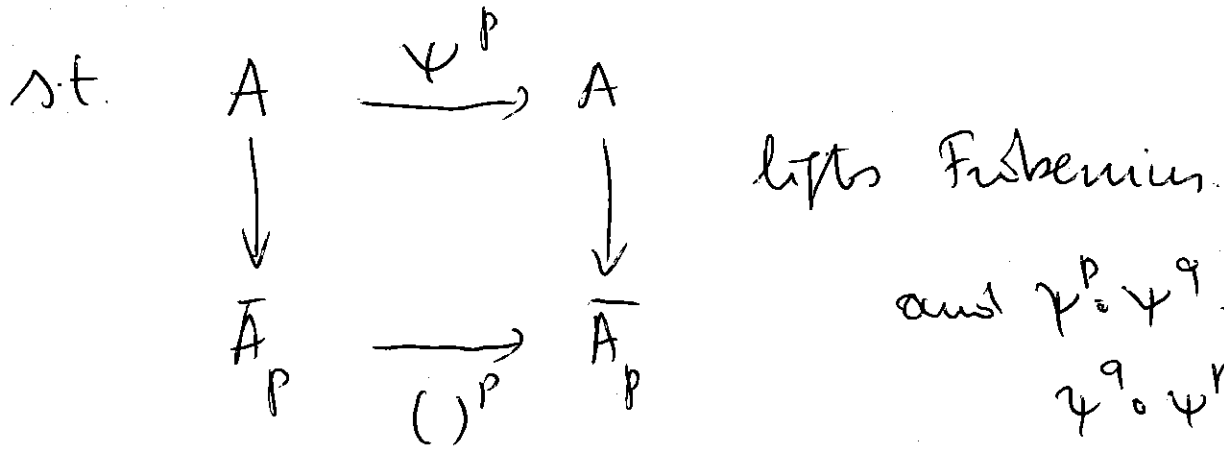
Fling = \mathbb{Z} -Ring + Frobenius lift

A \mathbb{Z} -algebra without additive torsion
 i.e. $n \cdot a = 0 \Rightarrow a = 0$ So no specific
 primes are more important to A than
 others

$$\bar{A}_p = A \otimes_{\mathbb{Z}} \mathbb{F}_p \text{ is } \mathbb{F}_p\text{-algebra}$$

so: $(-)^p$: $\bar{A}_p \rightarrow \bar{A}_p$ is \mathbb{F}_p -algebra
 morphism
 p-Frobenius morphism.

Demand $\exists \psi^p: A \rightarrow A$ \mathbb{Z} -algebra
 endomorphism



and $\psi^p \circ \psi^q = \psi^q \circ \psi^p$

\forall primes
 p, q

The define $\psi^n = \psi^{p_1^{e_1}} \circ \dots \circ \psi^{p_k^{e_k}}$

So have $\mathbb{N}_{+,x}$ family of endomorphisms

$$\psi^N : A \rightarrow A \quad (\text{sometimes called Adams operation})$$

Example:

① \mathbb{Z} take $\psi^N = \text{id}_{\mathbb{Z}}$

for every prime p

$$\begin{array}{ccc} \mathbb{Z} & \xrightarrow{\text{id}} & \mathbb{Z} \\ \downarrow & & \downarrow \\ \mathbb{F}_p & \xrightarrow{(\)^p = \text{id}} & \mathbb{F}_p \end{array}$$

② $\mathbb{Z}[t]$ take $\psi^N : t \mapsto t^N$

$$\begin{array}{ccc} \mathbb{Z}[t] & \xrightarrow{\psi^p} & \mathbb{Z}[t] \\ \downarrow & & \downarrow \\ \mathbb{F}_p[t] & \xrightarrow{(\)^p} & \mathbb{F}_p[t] \end{array}$$

Generalizes to Monoid algebras

"identity of \mathbb{F}_p
 $e \mapsto e \mapsto e^p$

More history
Hopf algebras
FL-structure
on $\mathbb{Z}[t]$

DETAILS!

Reality
the prop G
only considering
have G-structure
eg: $B \rightarrow B$
on L
with G-structure
Question
disc
non
take

Chebyshev line

$\mathbb{Z}[t]$ Fluz ω

$\mathbb{Z}[t, t^{-1}]$ Fluz ω

involution $t \leftrightarrow t^{-1}$ commutes with
Fl. structure i.e.

$$(\psi^n(t))^{-1} = (t^n)^{-1} = t^{-n} = \psi(t^{-1})$$

Das inherits invariant ω

$\mathbb{Z}[t, t^{-1}]^{\text{flip}}$ Fluz-structure.

behave if

$$a = a^{\text{flip}} \Rightarrow$$

$$\psi^n(a) = \psi^n(a^{\text{flip}}) = \psi^n(a)^{\text{flip}}$$

$$\mathbb{Z}[t, t^{-1}]^{\text{flip}} = \mathbb{Z}[x] \quad \text{with } x = t + t^{-1}$$

er kührende Fibonacci $t^2 + t^{-2}$ etc

$$\psi^2(x) = t^2 + t^{-2} = (t + t^{-1})^2 - 2 = x^2 - 2$$

$$\psi^3(x) = t^3 + t^{-3} = (t + t^{-1})^3 - 3(t + t^{-1}) = x^3 - 3x \text{ etc}$$

$\psi^n(x)$ zijn de Chebyshev polynomen.

CLAUWENS

enige Flang structure op $\mathbb{Z}[t]$ zijn

① tourische $t \rightarrow t^n$

② Chebyshev $t \rightarrow n$ -th Chebyshev

should be important!

Representation Theory

More motivating examples of Flurps.

G finite group

know # irred. reps = # conj. classes

character table

G	conj. class
χ_1	$\chi_i(g) = \text{Tr}(g \cdot V_i)$
...	
χ_n	

Example:

S_3	1	3	2
	()	(12)	(1,2,3)
$\chi_1 = T$	1	1	1
$\chi_2 = S$	1	-1	1
χ_3	2	0	-1

Representation Ring $R(G) = \mathbb{Z}\chi_1 + \dots + \mathbb{Z}\chi_n$

- + : componentwise contains all characters χ_V
- : induced by χ

$$\chi_i \cdot \chi_j = \chi_{V_i \otimes V_j}$$

is algebra mod $1 = \chi_T = \chi_1$

Vermeidung multistufig von S_3

•	x_1	x_2	x_3
x_1	x_1	x_2	x_3
x_2	x_2	x_1	x_3
x_3	x_3	x_3	$x_1+x_2+x_3$

$$R(S_3) = \mathbb{Z}[x, y] / (x^2 - 1, xy - y, y^2 - x - y - 1)$$

$$\cong \frac{\mathbb{Q}[x]}{x-1} \times \frac{\mathbb{Q}[x][y]}{(x+1)y} \times \frac{\mathbb{Q}[x][y]}{(x+1)(y-1)} \stackrel{(x-1)y \quad y(y-1)=x+1}{=} \mathbb{Q} \times \mathbb{Q} \times \mathbb{Q}$$

Extra structure: Adams operation

$\psi^n(x)$ is again class function

$$\psi^n(x)(g) \stackrel{\text{def}}{=} x(g^n)$$

Remark
 $\psi^{nm} = \psi^n \circ \psi^m = \psi^m \circ \psi^n$
 so suffices to

Example S_3

study ψ^p
 p prime

		()	(1, 2)	(1, 2, 3)
all n	ψ^n	()		
	ψ^{odd}		(1, 2)	
	ψ^{even}		()	
	ψ^{3h}			()
	$\psi^{\neq 3h}$			(1, 2, 3)

$$\text{b) } \psi^n(x_1) = x_1 \quad \forall n$$

$$\begin{cases} \psi^{\text{odd}}(x_2) = x_2 \\ \psi^{\text{even}}(x_2) = x_1 \end{cases}$$

$$\begin{aligned} \psi^2(x_3) &= (2, 1, -1) \\ &= x_1 + x_3 - x_2 \end{aligned}$$

$$\begin{aligned} \psi^3(x_3) &= (2, 0, 2) \\ &= x_1 + x_2 \end{aligned}$$

$$\psi^{p \neq 3}(x_3) = (2, 0, -1) = x_3$$

Compare this to powers of x_i in $R(S_3)$

$$x_1^n = x_1 \quad \forall n$$

$$x_2^n = \begin{cases} x_1 & n \text{ even} \\ x_2 & n \text{ odd} \end{cases}$$

$$x_3^2 = (4, 0, 1) = x_1 + x_2 + x_3$$

$$x_3^3 = (8, 0, -1) = 3x_3 + x_1 + x_2$$

$$x_3^p = (2^p, 0, -1)$$

$$2a + 2b = 2^p$$

$$-a + 2b = -1$$

$$\frac{\text{add eqs.}}{2 \quad 1}$$

~~$$4b = 2^p - 1$$~~

$$3a = 2^p + 1$$

$$a = \frac{2^p + 1}{3}$$

$$b = \frac{2^p - 2}{2} = 2^{p-1} - 1$$

$$x_3^p = \frac{2^p + 1}{3} x_3 + (2^{p-1} - 1)(x_1 + x_2)$$

$$= x_3 + \underbrace{(2^{p-1} - 1)}_{p \mid \quad} (x_1 + x_2 + x_3)$$

$p \mid$

(Kleine Fermat)

Dies hebbe $\forall p, \forall x$

$$\psi^p(x) - x^p \in pR(\mathbb{E})$$

das ψ^p in Frobenius liegt.

Voor algemene groep G

- ① Waarom is $\chi^n : R(G) \rightarrow R(G)$?
(er vs met λ - ν)
- ② Waarom is $\forall \alpha : \chi^P(\alpha) - \alpha^P \in p R(G)$?
voor p prime

① λ -invariantie van $R(G)$

$\forall n$ hebbe operatie $\lambda^n : \chi_V \rightarrow \chi_{\wedge^n V}$
 $\wedge^n V$ n -de uitwendig product van $\text{rep } V$

Als V basis $\{v_1, \dots, v_d\}$ heeft dan is

$\wedge^n V$ vectorruimte met basis

$$v_{i_1} \wedge v_{i_2} \wedge \dots \wedge v_{i_n}$$

met $1 \leq i_1 < i_2 < \dots < i_n \leq d$

G -actie op V heest uit tot G -actie op $\wedge^n V$
 diagonaal

$g \in G$ er kies basis $\{v_1, \dots, v_d\}$ zodat $g \cdot v$ diagonaal =

$$\begin{pmatrix} a_1 & & & 0 \\ & \ddots & & \\ & & \ddots & \\ 0 & & & a_d \end{pmatrix} \quad \text{dus} \quad \chi_V(g) = \sum_{i=1}^d a_i$$

Dan is

$$\chi_{\wedge^n V}(g) = \sum_{1 \leq i_1 < i_2 < \dots < i_n \leq d} a_{i_1} a_{i_2} \dots a_{i_n}$$

Adams-operations en λ -structure zijn relatief

CLAIM: $\forall n, \forall \chi$ geldt:

$$n \lambda^n(\chi) = \underbrace{\sum_{k=1}^n (-1)^{k-1} \psi^k(\chi) \lambda^{n-k}(\chi)}_*$$

Bewijs: volstaat gelijkheid te bewijzen $\forall g \in G$

Merk op $\psi^k(\chi)(g) = \chi(g^k)$

dan als $\chi(g) = \sum_{i=1}^d a_i$ dan $\psi^k(\chi)(g) = \sum_{i=1}^d a_i^k$

Bekijk nu $*$ (g) =

$$\sum_{k=1}^n (-1)^{k-1} \left(\sum a_i^k \right) \left(\sum_{1 \leq i_1 < i_2 < \dots < i_{n-k} \leq d} a_{i_1} a_{i_2} \dots a_{i_{n-k}} \right)$$

Merk op dat voor $k > 1$ de term $a_{i_1} \dots a_j^k \dots a_{i_{n-k}}$ 2x voorkomt:

hier is $(\sum a_j^{k-1}) (\sum a_{i_1} \dots a_j \dots a_{i_{n-k}})$

en 1 keer i

$$\left(\sum a_j^k \right) \left(\sum a_{i_1} \dots a_{i_{n-k}} \right)$$

met verschuifd teken, dan die vollen afwissel
 weg. Blijft teken die overblijven i_1, \dots, i_n
 van a_1, a_2, \dots, a_n

$$\left(\sum a_i \right) \left(\sum a_{i_1} \dots a_{i_{n-1}} \right) \text{ er zo zijn er}$$

juist n, dus dit is n maal gelijk à $n \lambda^n(x)(g)$



Hieruit volgt per inductie dat alle $\psi^n(x) \in R(\mathcal{G})$

$$\psi^1(x) = x$$

$$- \psi^2(x) + \psi^1(x) \lambda^1(x) = 2 \lambda^2(x)$$

$$\psi^3(x) - \psi^2(x) \lambda^1(x) + \psi^1(x) \lambda^2(x) = 3 \lambda^3(x)$$

etc.

② weten als $\psi^n(x) \in R(\mathcal{G})$ en ook $\lambda^n \in R(\mathcal{G})$
 $= x \uparrow \downarrow \otimes n$

Voor elke $f \in \mathcal{G}$ hebben we

$$\psi^n(x)(g) = \sum_{i=1}^n a_i^n \quad \lambda^n(g) = \left(\sum_{i=1}^n a_i \right)^n$$

Dus voor $n=p$ men volgt uit
binomische formule dat

$$\psi^p(g) - \chi^p(g) \in p R(\mathbb{C}) \text{ en dus zijn}$$

id's $\{\psi^n\}$ een Frobenius lift voor $R(\mathbb{C})$.

CLAIM : $\beta(n, \#G) = 1$

$\Rightarrow \psi^n$ is een permutatie v.d. irreducibele
rep.

i.e. $\psi^n(\chi_i) = \chi_j$ en $i \rightarrow j$ is permutatie

Bewijs

$$\begin{aligned} \langle \psi^n(\chi_i), \psi^n(\chi_i) \rangle &= \frac{1}{\#G} \sum_{g \in G} \psi^n(\chi_i)(g) \overline{\psi^n(\chi_i)(g)} \\ &= \frac{1}{\#G} \sum_{g \in G} \chi_i(g^n) \overline{\chi_i(g^{-n})} \end{aligned}$$

nu als $(n, \#G) = 1$ dan is $G \rightarrow G$ automorfisme
du $g \mapsto g^n$

$$= \frac{1}{\#G} \sum_{g^n \in G} \chi_i(g^n) \overline{\chi_i(g^{-n})} = \langle \chi_i, \chi_i \rangle = 1$$

Da $\psi^n(x_i) = x_j$ en het is een
permutatie want

$$\exists a: n \cdot a \equiv 1 \pmod{\#G} \quad \text{en}$$

$$\psi^{\#G}(x) = x \quad \text{en} \quad \psi^x(x) = \psi^y(x) \text{ als} \\ x \equiv y \pmod{\#G}$$

$$\text{dan } \psi^n(x) \circ \psi^a(x) = x$$

$$\text{dan } \psi^{\#G}(\psi^a(x)) = x. \quad \square$$

D_8 e Q_8 hebben zelfde karaktertabel, maar $R(D_8) \neq R(Q_8)$ als λ -rnf

(1)

Q_8	1	a^2	$\{a, a^3\}$	$\{b, a^2b\}$	$\{ab, a^3b\}$
D_8	1	a^2	$\{a, a^3\}$	$\{b, a^2b\}$	$\{ab, a^3b\}$
χ_1	1	1	1	1	1
χ_2	1	1	1	-1	-1
χ_3	1	1	-1	1	-1
χ_4	1	1	-1	-1	1
χ_5	2	-2	0	0	0

$Q_8 = \langle a, b \mid a^4 = 1 \quad b^2 = a^2 \quad b^{-1}ab = a^3 \rangle$

$D_8 = \langle a, b \mid a^4 = b^2 = 1 \quad b^{-1}ab = a^3 \rangle$

ψ_2	1	a^2	a	b	ab
Q_8	1	1	a^2	a^2	a^2
D_8	1	1	a^2	1	a^2

$\frac{abab}{ba^3}$

$ab^3 = 1$

$$\psi_2(x_1) = \frac{1 \quad 1 \quad 1 \quad 1 \quad 1}{1 \quad 1 \quad 1 \quad 1 \quad 1} = x_1$$

$$\psi_2(x_2) = \frac{1 \quad 1 \quad 1 \quad 1 \quad 1}{1 \quad 1 \quad 1 \quad 1 \quad 1} = x_1$$

● $\psi_2(x_3) = \text{_____} = x_1$

$\psi_2(x_4) = \text{_____} = x_1$

$$\psi_2(x_5) = \frac{2 \quad 2 \quad -2 \quad -2 \quad -2}{2 \quad 2 \quad -2 \quad 2 \quad +2} \begin{matrix} Q_8 \\ D_8 \end{matrix}$$

$$x_5^2 = 44000 = x_1 + x_2 + x_3 + x_4$$

Q_8
 $\psi_2(x_5) = x_1 + x_2 + x_3 + x_4 - 2x_1$
 $\psi_2^{D_8}(x_5) = x_1 + x_2 + x_3 + x_4 - 2x_2$

+2 +2 +2 -2 -2

