Stripping off structure

Aliens and Reality Question

$k \subset L$ finite Galois with group G

want to study k-algebras by only considering L-algebras + extra structure.

B L-algebra is said to have <u>G-structure</u> iff

$\forall g \in G$ $\exists$ automorphism $\varphi_g : B \to B$ extending Galois action by g on L

$$\begin{array}{ccc} B & \xrightarrow{\varphi_G} & B \\ \uparrow & & \uparrow \\ L & \xrightarrow[\sigma_g]{} & L \end{array}$$

B, B' L-algebras with G-structure

$B \xrightarrow{\varphi} B'$ G-morphism φ

~~$\circledast$~~ $\mathbb{A}$-algebra morphism preserving G-structure

$\varphi(g \cdot b) = g \cdot \varphi(b)$

<u>CLAIM</u>: L-algebras with G-structure and G-morphisms

$\Updownarrow$ equivalent

k-algebras

$$k\text{-algebras} \underset{B^G}{\overset{-\otimes_k L}{\rightleftarrows}} L\text{-algebras with } G\text{-structure}$$

$B = A \otimes_k L$ has G-structure via

$$\varphi_g(a \otimes \ell) = a \otimes \sigma_g(\ell)$$

$$\begin{array}{ccc} A & & A \otimes_k L \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f\otimes 1} \\ A' & & A' \otimes_k L \end{array}$$

$$(A \otimes_k L)^G = A \otimes 1$$

$$B^G \otimes_k L \longrightarrow B$$

$$a \otimes \ell \mapsto a\ell$$

Remarks

① Not every L-algebra has G-structure

Ex: $\mathbb{R} \subset \mathbb{C}$ e.g. $\mathbb{C}[x]/(x^2 - c)$ if $c \notin \mathbb{R}$

② L-algebras can have different G-structures

Ex: $\mathbb{R} \subset \mathbb{C}$ $\mathbb{C} \times \mathbb{C}$ 1) $(a,b) \mapsto (\bar{a}, \bar{b})$ $(\mathbb{R}\times\mathbb{R} \otimes \mathbb{C}$

2) $(a,b) \mapsto (\bar{b}, \bar{a})$ $(\mathbb{C} \otimes \mathbb{C}$

If we only consider L-algebras the functor

$$k\text{-alg} \longrightarrow L\text{-alg} \qquad A \to A \otimes_k L$$

can be viewed as "stripping" off G-structure.

If A is k-algebra
B is L-algebra

Have natural iso

$$\mathrm{Alg}_k(A, B|_k) = \mathrm{Alg}_L(A \otimes_k L, B)$$

$$f \mapsto (a \otimes l \mapsto f(a)l)$$

This says that

$$\begin{tikzcd} k\text{-alg} \arrow[r, bend left, "- \otimes_k L"] & L\text{-alg} \arrow[l, bend left, "\text{forget } L\text{-structure}"] \end{tikzcd}$$

are adjoint functors : forget L-structure is right adjoint of $- \otimes_k L$ and $- \otimes_k L$ is left adjoint of forgetting L-structure.

<u>In general</u>

$$\begin{tikzcd} \mathcal{A} \arrow[r, bend left, "F"] & \mathcal{B} \arrow[l, bend left, "G"] \end{tikzcd}$$

(F, G) is adjoint pair of functors iff

$\forall A \in Ob(\mathcal{A}) \qquad \forall B \in Ob(\mathcal{B})$

$$Hom_{\mathcal{A}}(A, G(B)) = Hom_{\mathcal{B}}(F(A), B)$$

If only want to work with L-algebras

$$\begin{tikzcd} \text{L-algebra with G-structure} \arrow[r, bend left, "\text{Forget G-struct}"] & \text{L-algebras} \arrow[l, bend left, "-|_h \otimes_k L"] \end{tikzcd}$$

Allows to define geometry "under L" by only working with L-objects + extra structure.

Need to find right adjoint of "forget extra structure" to know what L-object is considered as 'k-object'

Flings

Fling = $\underbrace{\mathbb{Z}\text{-Ring}}$ + Frobenius lift

A $\mathbb{Z}$-algebra without additive torsion
i.e. $n.a = 0 \Rightarrow a = 0$ So no specific primes are more important to A than others

$\bar{A}_p = A \otimes_{\mathbb{Z}} \mathbb{F}_p$ is $\mathbb{F}_p$-algebra

So: $\underbrace{(-)^p}_{}: \bar{A}_p \to \bar{A}_p$ is $\mathbb{F}_p$-algebra morphism
p-Frobenius morph.

Demand $\exists\, \psi^p : A \to A$ $\mathbb{Z}$-algebra endomorphism

s.t.
$$\begin{tikzcd} A \arrow[r, "\psi^p"] \arrow[d] & A \arrow[d] \\ \bar{A}_p \arrow[r, "()^p"'] & \bar{A}_p \end{tikzcd}$$
lifts Frobenius

and $\psi^p \circ \psi^q = \psi^q \circ \psi^p$ $\forall$ primes p, q

Then define $\psi^n = \psi^{p_1^{e_1}} \circ \dots \circ \psi^{p_k^{e_k}}$

So have $\mathbb{N}_{+,\times}$ family of endomorphisms

$$\psi^N : A \to A$$ (sometimes called Adams operations)

<u>Example</u>:

① $\mathbb{Z}$ take $\psi^N = id_{\mathbb{Z}}$

for every prime p

$$\begin{array}{ccc} \mathbb{Z} & \xrightarrow{id} & \mathbb{Z} \\ \downarrow & & \downarrow \\ \mathbb{F}_p & \xrightarrow[\underset{= id}{(\ )^p}]{} & \mathbb{F}_p \end{array}$$

② $\mathbb{Z}[t]$ take $\psi^N : t \mapsto t^N$

$$\begin{array}{ccc} \mathbb{Z}[t] & \xrightarrow{\psi^p} & \mathbb{Z}[t] \\ \downarrow & & \\ \mathbb{F}_p[t] & \xrightarrow[(\ )^p]{} & \mathbb{F}_p[t] \end{array}$$

$(\ )^p$ = identity of $\mathbb{F}_p$ & $t \mapsto t^p$

Generalizes to Monoid algebras

Mara instead may consider Fl-structure on $\mathbb{Q}[t]$

DETAILS!

$\boxed{\text{Chebyscheff line}}$

$\mathbb{Z}[t]$ Fling ol

$\mathbb{Z}[t,t^{-1}]$ Fling en

involutie $t \longleftrightarrow t^{-1}$ commutes with

Fl- structure i.e.

$$(\psi^n(t))^{-1} = (t^n)^{-1} = t^{-n} = \psi(t^{-1})$$

Dus inherits invariant ring

$\mathbb{Z}[t,t^{-1}]^{flip}$ Fling-structure.

because if

$$a = a^{flip} \Rightarrow$$

$$\psi^n(a) = \psi^n(a^{flip}) = \psi^n(a)^{flip}.$$

$\mathbb{Z}[t,t^{-1}]^{flip} = \mathbb{Z}[x]$ with $x = t+t^{-1}$

en bijhorende Frobenius lifts zijn dan

$$\psi^2(x) = t^2+t^{-2} = (t+t^{-1})^2 - 2 = x^2-2$$

$$\psi^3(x) = t^3+t^{-3} = (t+t^{-1})^3 - 3(t+t^{-1}) = x^3 - 3x \text{ etc}$$

$\psi^n(x)$ zijn de Chebycheff polynom.

$\boxed{\text{CLAUWENS}}$

enige F-ring structuren op $\mathbb{Z}[t]$ zijn

(1) torische $\quad t \to t^n$

(2) Chebycheff $\quad t \to$ n-th Chebych.

should be important!

Representation Rings

More motivating examples of Rings.

G finite group

Know $\#$ irred. rep $= \#$ conjug classes

character table

G	conj. class g
χ_1	
irred $\vdots$	$\chi_i(g) = \mathrm{Tr}(g\cdot|_{V_i})$
χ_n	

Example:

S_3	$()$	$(1,2)$	$(1,2,3)$
	1	3	2
$\chi_1 = T$	1	1	1
$\chi_2 = S$	1	-1	1
χ_3	2	0	-1

Representation Ring $R(G) = \mathbb{Z}\chi_1 + \cdots + \mathbb{Z}\chi_k$

$+$: componentwise contains all characters χ_V

$\cdot$: induced by

$$\chi_i \cdot \chi_j = \chi_{V_i \otimes V_j}$$

is algebra wrt $1 = \chi_T = \chi_1$

Vermenigvuldiging voor S_3

$\cdot$	χ_1	χ_2	χ_3
χ_1	χ_1	χ_2	χ_3
χ_2	χ_2	χ_1	χ_3
χ_3	χ_3	χ_3	$\chi_1+\chi_2+\chi_3$

$$R(S_3) = \mathbb{Z}[X,Y]/(X^2-1,\ XY-Y,\ Y^2-X-Y-1)$$

$(X-1)Y \qquad Y(Y-1)=X+1$

$$\subset \frac{\mathbb{Q}[X]}{X-1} \times \frac{\mathbb{Q}[X][Y]}{(X+1)Y} \times \frac{\mathbb{Q}[X][Y]}{(X+1)(Y-1)} = \mathbb{Q}\times\mathbb{Q}\times\mathbb{Q}$$

Extra structure: Adams operation

$\psi^n(\chi)$ is again class function

$$\psi^n(\chi)(g) \overset{\text{def}}{=} \chi(g^n)$$

Remark
$\psi^{nm} = \psi^n \circ \psi^m = \psi^m \circ \psi^n$
so suffices to study ψ^p
p prime

Example S_3

		$()$	$(1,2)$	$(1,2,3)$
all n	g^n	$()$		
	g^{odd}		$(1,2)$	
	g^{even}		$()$	
	g^{3k}			$()$
	$g^{\neq 3k}$			$(1,2,3)$

So $\psi^n(x_1) = x_1 \quad \forall n$

$$\begin{cases} \psi^{odd}(x_2) = x_2 \\ \psi^{even}(x_2) = x_1 \end{cases}$$

$$\psi^2(x_3) = (2, 1, -1)$$
$$= x_1 + x_3 - x_2$$

$$\psi^3(x_3) = (2, 0, 2)$$
$$= x_1 + x_2$$

$$\psi^{p \neq 3}(x_3) = (2, 0, -1) = x_3$$

Compare this to powers of x_i in $R(S_3)$

$x_1^n = x_1 \quad \forall n$

$$x_2^n = \begin{cases} x_1 & n \text{ even} \\ x_2 & n \text{ odd} \end{cases}$$

$$x_3^2 = (4, 0, 1) = x_1 + x_2 + x_3$$
$$x_3^3 = (8, 0, -1) = 3x_3 + x_1 + x_2$$

$$x_3^p = (2^p, 0, -1)$$

$$2a + 2b = 2^p$$

$$-a + 2b = -1$$

odd	even
2	1

$$3a = 2^p + 1$$

$$a = \frac{2^p+1}{3} \qquad b = \frac{2^p-2}{2} = 2^{p-1} - 1$$

$$x_3^p = \frac{2^p+1}{3} x_3 + (2^{p-1}-1)(x_1+x_2)$$

$$= x_3 + \underbrace{(2^{p-1}-1)}_{p \,\mid}(x_1+x_2+x_3)$$

(Kleine Fermat)

Dus hebben $\forall p, \forall x$

$$\psi^p(x) - x^p \in p\,R(G)$$

dus ψ^p is Frobenius lift.

$\boxed{\text{Voor algemene groep } G}$

(1) Waarom is $\psi^n : R(G) \to R(G)$?
(en vb met λ-ring)
(2) Waarom is $\forall \chi : \psi^p(\chi) - \chi^p \in pR(G)$?
voor p prie

(1) λ-ringen structuur of $R(G)$

$\forall n$ hebben operatie $\lambda^n : \chi_V \to \chi_{\underbrace{\Lambda^n V}}$

n-de uitwendig product v. repr V

Als V basis $\{v_1, \dots, v_d\}$ heeft dan is

$\Lambda^n V$ vectorruimte met basis

$$v_{i_1} \wedge v_{i_2} \wedge \dots \wedge v_{i_n}$$

$$\text{met } 1 \leq i_1 < i_2 < \dots < i_n \leq d$$

G-actie op V breidt uit tot G-actie op $\Lambda^n V$
diagonaal

$g \in G$ en kies basis $\{v_1, \dots, v_d\}$ zodat $g \cdot |_V$ diagonaal is

$$\begin{pmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_d \end{pmatrix} \quad \text{dus } \chi_V(g) = \sum_{i=1}^{d} a_i$$

Dan is

$$\chi_{\Lambda^n V}(g) = \sum_{1 \leq i_1 < i_2 < \cdots < i_n \leq d} a_{i_1} a_{i_2} \cdots a_{i_n}$$

Adams-operations en λ-structuur zijn gerelateerd

$\boxed{\text{CLAIM}}$: $\forall n$, $\forall \chi$ geldt:

$$n \lambda^n(\chi) = \underbrace{\sum_{k=1}^{n} (-1)^{k-1} \psi^k(\chi) \lambda^{n-k}(\chi)}_{*}$$

Bewijs. volstaat gelijkheid te bewijzen $\forall g \in G$

Merk op $\psi^k(\chi)(g) = \chi(g^k)$

dus als $\chi(g) = \sum_{i=1}^{d} a_i$ dan $\psi^k(\chi)(g) = \sum_{i=1}^{d} a_i^k$

Bekijk nu $*(g) =$

$$\sum_{k=1}^{n} (-1)^{k-1} \left(\sum a_i^k\right)\left(\sum_{1 \leq i_1 < i_2 < \cdots < i_{n-k} \leq d} a_{i_1} a_{i_2} \cdots a_{i_{n-k}}\right)$$

Merk op dat voor $k > 1$ de term $a_{i_1} \cdots a_j^k \cdots a_{i_{n-k}}$

2x voorkomt:

1 keer in $\left(\sum a_j^{k-1}\right)\left(\sum a_{i_1} \cdots a_j \cdots a_{i_{n-(k+1)}}\right)$

en 1 keer in

$$\left(\sum a_j^k\right)\left(\sum a_{i_1} \cdots \overset{j}{\vee} \cdots a_{i_{n-k}}\right)$$

met verschillend teken, dus die vallen allemaal weg. Enige terme die overblijven zijn vd ~~...~~ vorm $a_1 a_2 \cdots a_n$ in

$$\left(\sum a_i\right)\left(\sum a_{i_1} \cdots a_{i_{n-1}}\right)$$ en zo zijn er juist n, dus dit is idd gelijk à $n\,\lambda^n(x)(g)$

$\boxtimes$

Hieruit volgt per inductie dat alle $\psi^n(x) \in R(G)$

$$\psi^1(x) = x$$

$$-\psi^2(x) + \psi^1(x)\lambda^1(x) = 2\lambda^2(x)$$

$$\psi^3(x) - \psi^2(x)\lambda^1(x) + \psi^1(x)\lambda^2(x) = 3\lambda^3(x)$$

etc.

(2) weten als $\psi^n(x) \in R(G)$ en ook $x^n \in R(G)$

$= x^{\otimes n}$

Voor elke $g \in G$ hebben we

$$\psi^n(x)(g) = \sum_{i=1}^{d} a_i^n \qquad x^n(g) = \left(\sum_{i=1}^{d} a_i\right)^n$$

Dus voor $n = p$ priem volgt uit binomiaal formule dat

$$\psi^p(g) - \chi^p(g) \in p\, R(G)$$ en dus zijn

idd $\{\psi^n\}$ een Frobenius lift voor $R(G)$.

<u>CLAIM</u> : $(n, \# G) = 1$

$\Rightarrow \psi^n$ is een permutatie vd. irreducibele rep.

i.e. $\psi^n(\chi_i) = \chi_j$ en $i \to j$ is permutatie

<u>Bewijs</u>

$$\langle \psi^n(\chi_i), \psi^n(\chi_i) \rangle = \frac{1}{\# G} \sum_{g \in G} \psi^n(\chi_i)(g)\, \psi^n(\chi_i)(g^{-1})$$

$$= \frac{1}{\# G} \sum_{g \in G} \chi_i(g^n)\, \chi_i(g^{-n})$$

nu als $(n, \# G)$ dan is $\begin{array}{c} G \to G \\ g \mapsto g^n \end{array}$ automorfisme dus

$$= \frac{1}{\# G} \sum_{g^n \in G} \chi_i(g^n) \chi_i(g^{-n}) = \langle \chi_i, \chi_i \rangle = 1$$

Dus $\psi^n(x_i) = x_j$ en het is een permutatie want

$\exists a: \quad n \cdot a \equiv 1 \bmod \# G$ en

~~$\psi^{\#G}(x)$~~ en $\psi^x(x) = \psi^y(x)$ als $x \equiv y \bmod \# G$

dus $\psi^n(x) \circ \psi^a(x) = x$

dus $\psi^n(\psi^a(x)) = x$. $\boxtimes$

1

D_8 en Q_8 hebben zelfde character tabel, maar $R(D_8) \not\cong R(Q_8)$ als λ-ring

Q_8	1	a^2	$\{a, a^3\}$	$\{b, a^2b\}$	$\{ab, a^3b\}$
D_8	1	a^2	$\{a, a^3\}$	$\{b, a^2b\}$	$\{ab, a^3b\}$
χ_1	1	1	1	1	1
χ_2	1	1	1	-1	-1
χ_3	1	1	-1	1	-1
χ_4	1	1	-1	-1	1
χ_5	2	-2	0	0	0

$$Q_8 = \langle a, b \mid a^4 = 1 \quad b^2 = a^2 \quad b^{-1}ab = a^3 \rangle$$

$$D_8 = \langle a, b \mid a^4 = b^2 = 1 \quad b^{-1}ab = a^3 \rangle$$

Ψ_2	1	a^2	a	b	ab
Q_8	1	1	a^2	a^2	a^2
D_8	1	1	a^2	1	[illegible]

$\dfrac{ab\,ab}{ba^3}$

$ab\,\cancel{ba}\,a^3 = 1$

$$\psi_2(x_1) \quad \frac{1 \qquad 1 \qquad 1 \qquad 1 \qquad 1}{1 \qquad 1 \qquad 1 \qquad 1 \qquad 1} \quad = x_1$$

$$\psi_2(x_2) \quad \frac{1 \qquad 1 \qquad 1 \qquad 1 \qquad 1}{1 \qquad 1 \qquad 1 \qquad 1 \qquad 1} \quad = x_1$$

$$\psi_2(x_3) \quad \overline{\qquad\qquad\qquad\qquad\qquad} \quad = x_1$$

$$\psi_2(x_4) \quad \overline{\qquad\qquad\qquad\qquad\qquad} \quad = x_1$$

$$\psi_2(x_5) \quad \frac{2 \qquad 2 \qquad -2 \qquad -2 \qquad -2}{2 \qquad 2 \qquad -2 \qquad 2 \qquad +2} \quad \begin{matrix} Q_8 \\ D_8 \end{matrix}$$

$$x_5^2 = 4\ 4\ 0\ 0\ 0 = x_1 + x_2 + x_3 + x_4$$

$$\boxed{\begin{aligned} \psi_2^{Q_8}(x_5) &= x_1 + x_2 + x_3 + x_4 - 2x_1 \\ \psi_2^{D_8}(x_5) &= x_1 + x_2 + x_3 + x_4 - 2x_2 \end{aligned}}$$

$+2 \quad +2 \quad +2 \quad -2 \quad -2$