$\boxed{\text{New geometry coming from } \mathbb{F}_1}$

<u>successes</u> of $\mathbb{F}_1$-alg = λ-rings

① recover examples from other approach

② explaining $\overline{\mathbb{F}_1} = \mu_\infty$ (last time)

③ gives interesting new examples $R(G)$

④ explaining Smirnov's $\mathbb{P}^1_{\mathbb{F}_1} = \{[0], [1], [2], \dots, [\infty]\}$

⑤ allows to make sense of

$$\mathbb{P}^1_{\mathbb{F}_1} \times_{\mathbb{F}_1} \operatorname{Spec}(\mathbb{Z}) = \mathbb{P}^1_{\mathbb{Z}} \quad (\text{without } \lambda\text{-structure})$$

<u>failure</u> (?) Smirnov's graph is not $\mathbb{F}_1$-subscheme of $\mathbb{P}^1_{\mathbb{Z}}$.

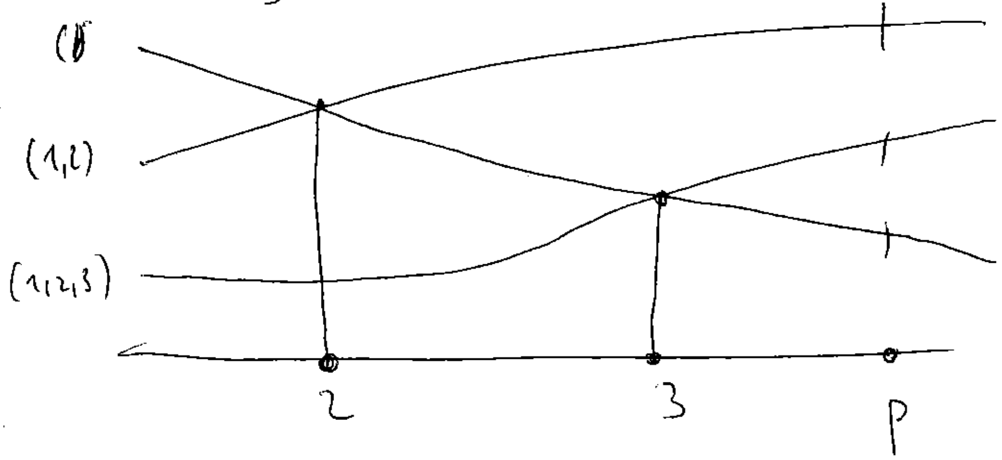
$\boxed{\text{Today}}$: what is new in $\mathbb{F}_1$-geometry

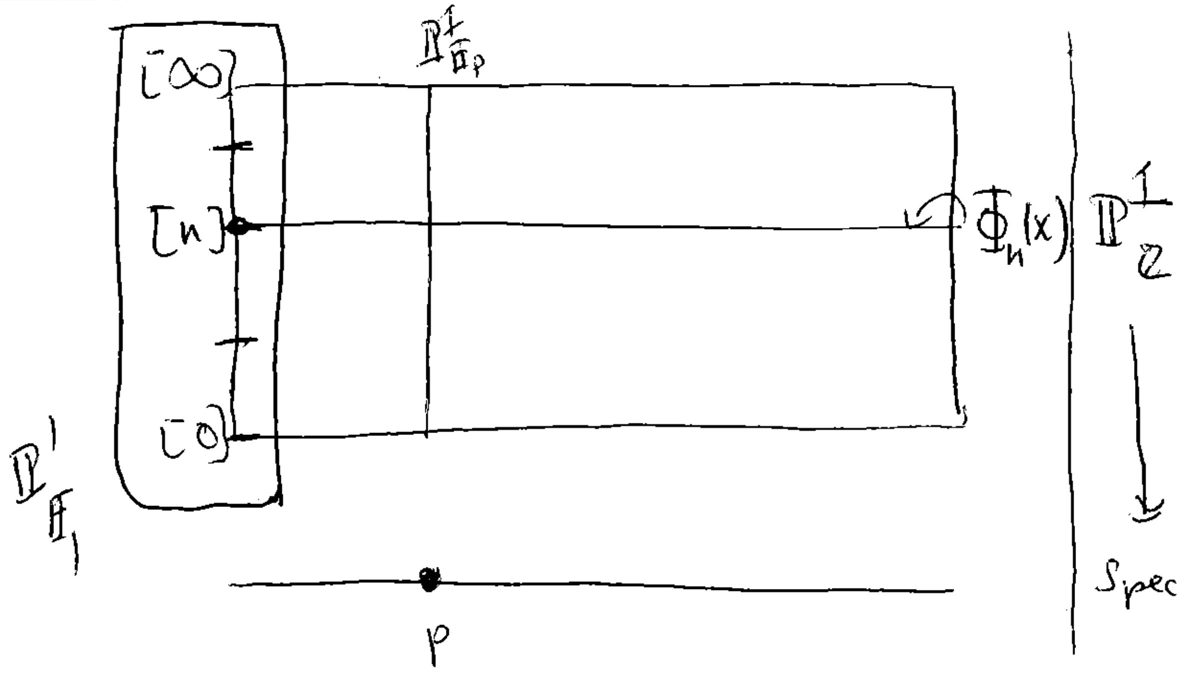
i.e. take λ-ring A what is different from usual approach $\operatorname{Spec}(A)$?

Main example: $\mathbb{P}^1_{\mathbb{Z}}$

Spec $R(G)$

$G = S_3$



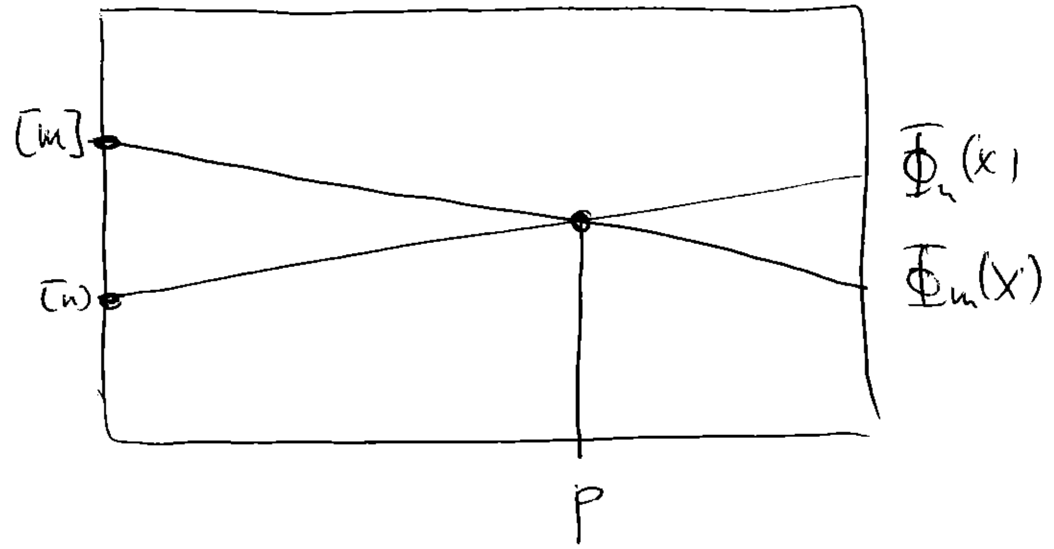


- on $\mathbb{P}^1_{\mathbb{Z}}$ toric λ-structure $\Rightarrow$ have seen that only subschemes preserving λ-structure come from roots of unity

  Therefore $\mathbb{P}^1_{\mathbb{F}_1} = \{[0], [1], [2], \ldots, [\infty]\}$

  with $[n] \rightsquigarrow \dfrac{\mathbb{Z}[x]}{(x^n-1)}$ or $\dfrac{\mathbb{Z}[x]}{(\Phi_n(x))}$

- <u>So</u> 'points" of $\mathbb{P}^1_{\mathbb{F}_1}$ determine 'horizontal' <u>sections connecting different fibers</u> $\mathbb{P}^1_{\mathbb{F}_p}$ [redacted] have [redacted]



criterium:

- If $\frac{m}{n} \neq p^k \Rightarrow \Phi_n(x)$ and $\Phi_m(x)$ comaximal

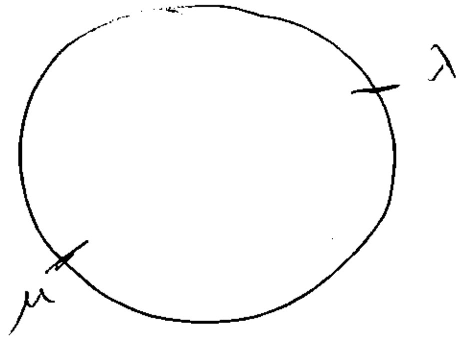
  If $\frac{m}{n} = p^k \Rightarrow \Phi_m(x) = \Phi_n(x)^d \mod p$ for some $d \Rightarrow$ not comaximal and intersect over p.

- This is new feature and says that points [m] and [n] share some geometric info or that [illegible] proved "close" in some [illegible] from Zouili) [illegible] = composite [illegible] ta. [illegible] th

$\lambda \leftrightarrow \mu$

$\lambda, \mu \in \mu_\infty$ are "close" iff

$\frac{\lambda}{\mu}$ has order p^k

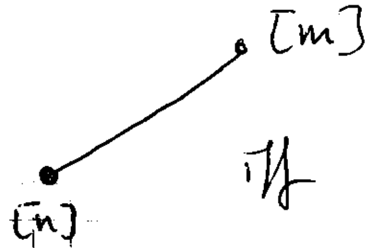


Relation $\lambda \leftrightarrow \mu$ is preserved under action of $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$

So instead of $\mathbb{P}^1_{\mathbb{F}_1} = \{[0], [\infty], [1], [2], \dots\}$

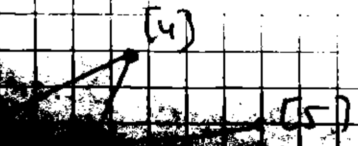
set with cofinite topology

now have graph.



iff $\frac{m}{n}$ or $\frac{n}{m} = p^l$

for some prime p



(?) … ally in order

To ge … ities ?

duality

ALGEBRA		GEOMETRY
coordinate ring of variety X $\mathcal{O}(X)$	$\leftarrow$	variety X
algebra A	$\longrightarrow$	maximal ideals of A

other duality: KOSTANT duality $\boxed{Alg(A,B) = Coalg(B^{\circ}, A^{\circ})}$

ALGEBRAS COALGEBRAS

$$A \longmapsto A^{\circ} = \left\{ \lambda : A \to \mathbb{C} \text{ linear s.t. } \exists I \triangleleft A \quad I \subset Ker(\lambda) \text{ and } A/I \text{ finite dm'l} \right\}$$

$$Hom_{\mathbb{C}}(C, \mathbb{C}) = C^{*} \longleftarrow\!\!| \; C$$

why different?

algebra $A \otimes A \xrightarrow{m} A$

dualize $A^{*} \xrightarrow{m^{*}} (A \otimes A)^{*} \neq A^{*} \otimes A^{*}$

in general

$= A$ is f.d.

(advantage : even works for noncomm algs)

What is geometric content of A° ?

How are A and $(A^\circ)^*$ related?

Example $A = \mathbb{C}[x]$ $\qquad \alpha \qquad \mathbb{A}^1_{\mathbb{C}}$

$I \triangleleft \mathbb{C}[x]$ $\qquad I = \prod (x-\alpha_i)^{n_i}$

$$\mathbb{C}[x]/I = \frac{\mathbb{C}[x]}{(x-\alpha_1)^{n_1}} \times \cdots \times \frac{\mathbb{C}[x]}{(x-\alpha_l)^{n_l}}$$

because ideals $x-\alpha_i$ are comaximal

$$A^\circ = \varinjlim \left(\left(\frac{\mathbb{C}[x]}{I}\right)^*\right)$$

$$\left(\mathbb{C}[x]/I\right)^* = \prod \left(\frac{\mathbb{C}[x]}{(x-\alpha_i)^{n_i}}\right)^*$$

So ... $\left(\frac{\mathbb{C}[y]}{(y^n)}\right)^* = \mathbb{C}1 + \mathbb{C}z + \dots + \mathbb{C}z^{n-1}$

Coal...

$\varepsilon(z^i) = \delta_{0i}$

ε is "evaluation @ pt"

$$\varinjlim_{n} \left(\frac{\mathbb{C}[y]}{(y^n)} \right)^{*} = \underbrace{U_{lie}(\mathbb{C}z)}$$

universal lie algebra of Abelian 1-dim'l lie algebra $\mathbb{C}z$

$$\boxed{A^{0} = \bigoplus_{p \in \mathbb{A}^1} U(T_{p,\mathbb{A}^1})}$$

- in general $\mathcal{O}(X)$ X affine <u>smooth</u> variety

$$\mathcal{O}(X)^{0} = \bigoplus_{x \in X} U(T_{X,x})$$

if X is <u>not</u> smooth $\Rightarrow$ $\mathcal{O}(X)^{0}$ is sub coalgebra of this

- How to recover X from $\mathcal{O}(X)^{0}$?

as "<u>coradical</u>" of $\mathcal{O}(X)^{0}$

simple subcoalgebras

X

$\mathcal{O}$

$$(A^\circ)^* = \Big(\bigoplus_{\alpha \in \mathbb{A}'} U(T_{\alpha,\mathbb{A}'}) \Big)^*$$

$$= \prod_{\alpha \in \mathbb{A}'} \mathbb{C}[[x-\alpha]]$$

Connection with A: algebra map

$$\mathbb{C}[x] \longrightarrow \prod_{\alpha \in \mathbb{A}'} \mathbb{C}[[x-\alpha]]$$

$$f \mapsto \prod f_\alpha$$

f_α is Taylor expansion of f in α

so $(A^\circ)^* = \prod$ (completion of stalks in structure sheaf)

or $\prod$ stalks in étale top.

<u>so</u> duality ALG $\rightleftarrows$ COALG

[illegible] and then [illegible]

Problem with λ-rgs: are $/\mathbb{Z}$

so do we have Kostant duality for $\mathbb{Z}$-rigs?

YES (more generally for Dedekind domain)

A $\mathbb{Z}$-Ring

define $A^\circ = \{ \lambda : A \to \mathbb{Z}$ ($\mathbb{Z}$-linear) : $\exists\, I \triangleleft A$, $I \subset \mathrm{Ker}(\lambda)$, A/I is f.g + torsion free (= f.g. projective) $\}$

again $\mathrm{Alg}_{\mathbb{Z}}(A, B) = \mathrm{Coalg}_{\mathbb{Z}}(B^\circ, A^\circ)$.

now what is special if A is λ-rig?

define

$A^\circ_{new} = \{ \lambda : A \to \mathbb{Z} \mid \exists\, J \triangleleft A$ $J \subset \mathrm{Ker}(\lambda)$

... torsion free $\}$

... A°)

Example $\mathbb{P}^1_{\mathbb{Z}}$ or $\mathbb{Z}[x]$

subschemes with λ-structure $= \mathbb{V}(X^n - 1)$

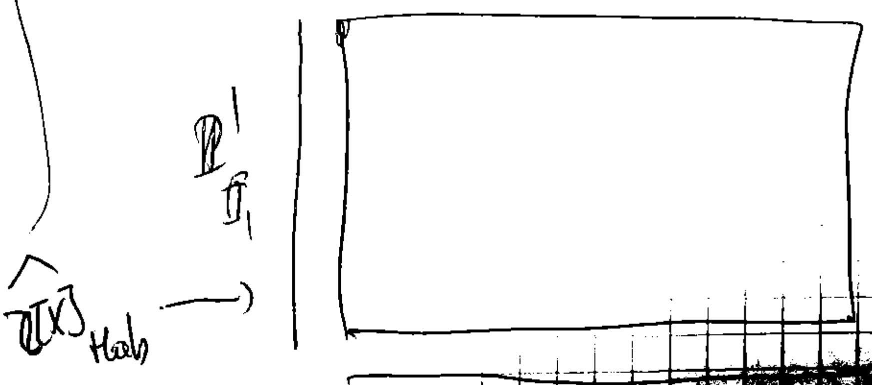
$$X^n - 1 = \prod_{d \mid n} \Phi_d(n)$$

$$\operatorname{dens}_{new} \mathbb{Z}[X]^{0*} = \varprojlim_{d,h} \left(\frac{\mathbb{Z}[x]}{\Phi_d(x)^h} \right)$$

$$= \varprojlim \frac{\mathbb{Z}[x]}{\underbrace{(x^n-1)(x^{n-1}-1)\cdots(x-1)}_{\text{"}[n]!\text{"}}}$$

in $\mathbb{P}^1_{\mathbb{F}_1}$ analogon

van profinite numbers $\hat{\mathbb{Z}}$



eles

$\Phi_p(x) < n$

Special fact about power series in $\hat{\mathbb{Z}}[x]_{\text{Hab}}$: are defined in all roots of unity! but usually diverge everywhere else

"functions leaking out of roots of unity"

New Topology and structure sheaf on $\mathbb{P}^1 = \mathbb{N} \cup \{\infty\}$

$S \subset \mathbb{N}$

Define Φ_S^* the multiplicative set in $\mathbb{Z}[x]$

generated by the $\Phi_s(x)$ for $s \in S$

For ... $\subset \mathbb{N}$ can define

... $\dfrac{\mathbb{Z}[x]}{f(x)}$

If ... $\prod \Phi_s(x)$-adic completion of $\mathbb{Z}[x]$

in ...

If $S' \subset S$ then have canonical $\mathbb{Z}$-morph

$$\rho^{S}_{S'} : \widehat{\mathbb{Z}[x]}_S \to \widehat{\mathbb{Z}[x]}_{S'}$$

and if $\mathbb{N}$ has <u>discrete topology</u> the

$(X, \widehat{\mathbb{Z}[x]}_S)$ is <u>pre</u>sheaf of rings

Want to know how to refine system of opens s.t. is a sheaf with global section $\widehat{\mathbb{Z}[x]}_{Hab}$

Habiro proved following facts

① If $S' \subset S$ s.t. $\forall n \in S$, $\exists n' \in S'$ and $\exists$ path $n' - n_1 - \cdots - n$ <u>lying entirely in S</u>

$\Rightarrow \rho^{S}_{S'} : \widehat{\mathbb{Z}[x]}_S \hookrightarrow \widehat{\mathbb{Z}[x]}_{S'}$ is injective

② If S is <u>saturated</u> (i.e. if $n \in S \Rightarrow \{d : d \mid n\} \subset S$ then

$\widehat{\mathbb{Z}[x]}$

$\Phi_n(x)$

hs"

Make sense to take topology on $\mathbb{P}'$ with opens the saturated subsets

$S_1 \cap S_2$ again saturated

$\cup S_i$ "

$\emptyset$ sat

$\mathbb{N}$ sat.

$\Rightarrow$ (S sat, $\widehat{\mathbb{Z}[x]}_S$) is sheaf of rings with $\Gamma = \widehat{\mathbb{Z}[x]}_{Hab}$.

$[n] \text{ — } [m] \qquad m \geq n$

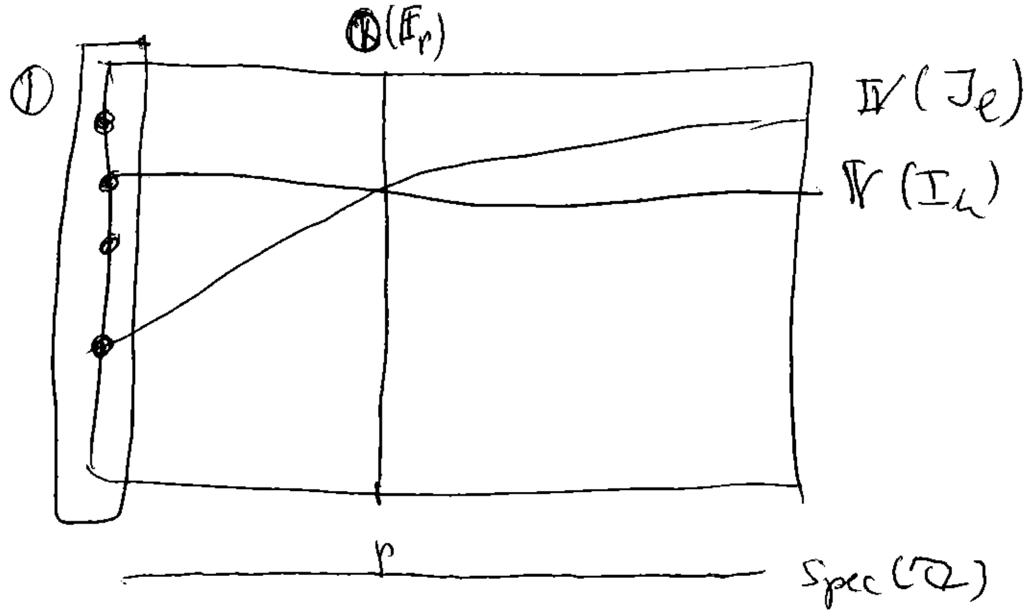
$\Downarrow$

every open containing m contains n

[illegible] is saturated

How to extend this to general λ-rings

A an object over $\mathbb{F}_1$



"pts" $\longleftrightarrow$ irreducible components of $\mathbb{V}(I)$ for all I s.t. A/I is λ-ring + torsion free and f.g. $/\mathbb{Z}$

$\downarrow$

when reduced we can classify these via Galois sets of $\mathbb{F}_1$

$\updownarrow$

Really [illegible] (A°)

[illegible] $\mathbb{V}(I_1) \cup \mathbb{V}(I_2) \cup \dots$

[illegible] $\mathbb{V}(I_k) \cap \mathbb{V}(J_\ell) \neq \emptyset$ i.e.

[illegible] intersect over certain p's

$\Rightarrow$ pt $\mathbb{V}(I_k)$ [illegible] (J_ℓ) connected in p-tree of A

<u>again</u> : look at completion

s.t. $(A^0)^* = \hat{A}_{\mathbb{N}}$

and define saturated topology etc.

important question !

what is all this for $A = W(\mathbb{Z})$????

<u>to begin</u>. first ~~quaso~~ λ-quotients of $W(\mathbb{Z})$ which are torsion free and f.g.