1 : Thanks to Gunther for gently pushing me to give this talk. As you see the stress should be on algebraic rather than on D-branes. What we will do is take a naive algebraic view on Polchinski's interpretation of D-branes, drop all physics from it and what remains is still an interesting question on algebra maps to Azumaya algebras, a natural generalization of classical representation theory of algebras.

I will use this problem as an excuse to recall some results in a particular brand of noncommutative algebraic geometry based on representation schemes, and try to explain why we do the things we do the way we do them.

Perhaps i should stress that all my algebras have a unit element and are defined over the complex numbers and all are finitely generated. So a typical commutative algebra would be the coordinate ring of an affine variety but we allow noncommutative algebras such as path algebras of quivers and their quotients.

2 : The algebras we will associate to D-branes are Azumaya algebras which are locally in the etale or analytic topology full matrixrings over the center. If the size of the matrices is n we will say that A is an Azumaya algebra of degree n.

Azumaya algebras with the same center form a symmetric monoidal category inder tensor-product and the Morita equivalence classes of such algebras form an Abelian group called the Brauer group of the center, which can be computed and is an important geometric invariant.

The bimodules of an Azumaya algebra are equivalent to modules over the center and as most noncommutative geometric constructions are made out of bimodules, the noncommutative geometry of Azumaya algebras is therefore expected to be almost the same as the central geometry.

A consequence is that maps from Azumaya algebras are fully determined by their image and the centralizer of the image, so R^A denotes all elements of R commuting with the image of A. If R is also an Azumaya algebra so is this centralizer ring which is a form of the double centralizer theorem.

3: Let's give some examples which will be familiar to most of you. Take the 2-dimensional torus with coordinate ring the Laurant polynmials in s and t and consider the quantum-torus at an n-th root of unity, then this is an Azumaya algebra of degree n over teh torus and its class in the Brauer group has order n.

If we consider these algebras for all n, they will generate the Brauer group of the torus which is isomorphic to $\mathbb{Q}/\mathbb{Z}$. So, in general the Brauer group is pretty big and for toric varieties their exists a combinatorial description of it.

4 : Now let's turn to Polchinski's interpretation of D-branes which are (as you all know far better than i do) boundary conditions for open string states. If you have just one D-brane then it is equipped with a $U(1)$-gauge field and to an algebraic geometer would see the brane just as a subvariety of space-time.

But if you stack n branes on the same subvariety then there is a gauge enhancement to $U(n)$, the reason being that there are now massless string states between the different sheets and they behave like the matrix-elements E_{ij}. So this stack of n D-branes becomes a noncommutative object which locally looks like $n \times n$ matrices, that is an Azumaya algebra.

So purely in algebraic terms, if you have 1 brane its inclusion in space time is determined by the epimorphism between teh coordinaterings restricting regular function of space-time to the brane.

However, if you have a stack of n branes then the inclusion in space-time gives an algebra morphism from the coordinate ring of space-time to this noncommutative object which is an Azumaya algebra over your subvariety X.

5 : In analogy with commutative algebraic geometry one would like to assign noncommutative geometric objects to the Azumaya algebra (and possibly to the space-time Y) and a map describing the embedding which allows one to reconstruct from it the given algebra map.

Liu and Yau tried to address this problem in a neverending series of papers on the arxiv and they called their approach 'Azumaya noncommutatiev geometry'

But they quickly ran into problems well-known to ringtheorists for over 40 years namely that most proposals to assign a prime-spectrum and structure sheaf to a noncommutative algebra do not behave functorially.

Let's consider the easiest of situations, that of an algebraic D-brane on the affine line, or in other words an n-dimensional representation of the polynomial ring in one variable. Now in all dialects in noncommutative geometry one would associate to complex $n \times n$ matrices as space just one point. But there is no point on the affine line containing enough information to reconstruct the representation which is given by sending x to a complex matrix. So one would assume that this map determines a degree n subscheme of teh affine line containing information about the eigenvalues of the matrix and their multiplicities. So we would rather expect the noncommutative gadget corresponding to $\mathbb{C}[x]$ able to describe all embeddings of D0-branes to be something like the Hilbert scheme, so something n-dimensional.

The main problem we will address is what the appropriate space should be for more complicated space-times and more complicated Azumaya algebras.

6 : So how does noncommutative algebraic geometry deals with this failure of functoriality. Well, to be honest we cheat a little bit. Instead of assigning some prime spectrum to an algebra we associate to it something that is obviously functorial with respect to algebra maps. Some take this object to be the category of modules or its derived category, we will take a somewhat smaller object namely teh category of all finite dimensional representations of the algebra. Restriction of scalars then gives a map between these objects which is a functor between the categories. Clearly the game is then to determine how much information of the algebra we can recover from this so called geometric object.

So, are we doing just category theory? Well not quite, we can approximate the full category by looking at the set of all n-dimensional representations which we will see in a moment is an ordinary commutative variety, so we can view our noncommutative geometric object as a limit of commutative varieties.

Isomorphism or representations gives an action of PGL_n on this variety and a program started by Artin and Procesi a long time ago was to use the tools of Mumford's geometric invariant theory in order to study these level n approximations of our object.

So, are we just doing equivariant commutative geometry? Not quite. First of all there are plenty of PGL_n-varieties which are not representation varieties and we are not interested in all equivariant data, but only the data induced from noncommutative geometric gadgets defined on R. So, for example, derivations of R should induce equivariant vector-fields on all these representation varieties.

So let us define all this more carefully and see what the main results are in this outskirt of noncommutative algebraic geometry.

7 : An unconventional way to describe these representation varieties is by using an old idea of George Bergman. Define the n-th root of an algebra R to be the centralizer of complex $n \times n$ matrices in the free product of R with it. Now because $M_n(C)$ is an Azumaya algebra we can recover this free product by tensoring this centralizer ring with $M_n(\mathbb{C})$.

Then it follows quickly from universal properties of these rings that this n-th root algebra represents the representation functor on all algebras, by this we may that for any algebra B the set of algebra maps from the n-th root of R to B is equal to the set the functor associates to B, in this case the set of all algebra maps from R to full matrices over B.

But this is still all rather formal noncommutative algebra and we would like to have genuine commutative geometric objects giving level n approximations of this full noncommutative thing.

8 : Grothendieck's approach to algebraic geometry was to stress the importance of representatble functors from commutative algebras to sets. Such a functor is called an affine scheme if there is an affine commutative algebra $\mathbb{C}[X]$ called the coordinate ring of the variety representing this functor.

Consider the functor which assigns to a commutative algebra C the set of all algebra maps from R to $n \times n$ matrices over C. For example, the complex points are just the n-dimensional representations of R and that's why we call this the n-th representation scheme and using Bergman's result we see that it is represented by the abelinization of the n-th root of R, that is we have that the coordinate ring of the representation scheme is just the abelianized n-th root algebra.

9 : But what are the advantages of this wildly universal algebra approach?

First we can define a universal map j_n allowing us to view elements of R as $n \times n$ matrices over this coordinate ring and this map will be important later on.

Secondly it gives us a way to describe the PGL_n-action on the representation scheme coming from the usual action by conjugation on complex matrices and using representability and then abelianization.

But perhaps most importantly we can mimick this construction for other categories of algebras in which we can form free products and define commutators. For example if you consider super-algebras you get this way a super-representation scheme, if you take graded algebras you get graded representation schemes and if you take differential graded algebras you can define in this way the derived representation schemes on which Yuri Berest will talk on wednesday.

But let us return to these ordinary representation schemes and recall the most important results about them.

10 : Ideally if you have a scheme with a group action you would like to describe the orbits, in our case the isomorphism classes of n-dimensional representations.

Now, Geometric invariant theory tells us that this is not always possible and that the best approximation to the non-existent orbit-space is the quotient scheme with coordinate ring the invariant polynomial functions and that this quotient scheme parametrizes the closed orbits.

Artin proved that in the case of representation schemes the closed orbits are precisely the isomorphism classes of semi-simple n-dimensional representations and therefore that the quotient map π assigns to a representation the direct sum of its Jordan-Hölder simple factors.

If your scheme is a principal bundle then of course all orbits are closed and the quotient scheme is really the orbit space. Artin classified the principal PGL_n-bundles over the prime spectrum of a commutative ring as the representation schemes of Azumaya algebras of degree n over C.

Moreover, in this case one recovers the center from the representation scheme as the invariant polynomial functions and the Azumaya algebra itself as the ring of all equivariant maps from the representation scheme to the affine space of all complex matrices equipped with the PGL_n-action by conjugation.

Although there is no hope to reconstruct more general noncommutative algebras R just from their n-dimensional representations, Artin conjectured that one should be able to describe explicitly the ring of equivariant maps and consider it as the best level n approximation of R and he also conjectured that in general the invariant polynomials shoudl be generated by traces.

11 : These conjectures were proved a couple of years later by Claudio Procesi. Recall the universal map j_n sending elements of R to $n \times n$ matrices over the coordinate ring of te representation scheme, so we can take traces of these matrices and they are obviously invariant polynomial maps. Procesi proved that they indeed generate all invariants.

Moreover, he proved that the ring of all equivariant maps is generated by the images of teh universal map j_n together with all the traces and in fact he was able to describe the kernel of this map as being generated by all formal Cayley-Hamilton identities of degree n.

These results also illustrate Kontsevich's philosophy that all relevant equivariant data (such as invariants and equivariant maps) is really induced on all representation schemes by objects defined on the noncommutative level. For example one could define noncommutative functions to be this vectorspace or its symmetric algebra because they determine at every level n the correct equivariant functions namely teh invariants.

Also note that whereas these construction were meant to be applied to noncommutative algebras R we can also apply them to commutative rings and in this case the relevant algebra describing equivariant maps is again a commutatiev algebra namely the tensorproduct of R with the symmetric algebra over it.

12 : Kontsevich gave some examples of his philosophy such as interpreting Procesi's result as noncommutative functions giving the invariants or derivations of R giving equivariant vector-fields but since a few years we have a general procedure to implement this idea by what i call Michel's Machine because it is due to Michel Van den Bergh.

By our universal map j_n we can view the matrix-ring as an R-bimodule as well as a module over the coordinate ring via the diagonal embedding and so tensoring with this matrix ring gives a nice functor from R-bimodules to sheaves over the representation scheme and hence we can induce all geometric gadgets defined in terms of bimodules, for example deRham-complexes and stuff like that to get equivariant complexes of sheaves on all representation schemes.

We will not use this functor but I guess Yuri Berest will tell you more about it in the case of derived representation schemes on wednesday.

13 : Clearly one expects all this to work best for noncommutative analogoins of manifolds which are what Kontsevich called formally smooth algebras as they have the property that all of its representation schemes are smooth and I've written a thick book giving explicit etale local descriptions of its level n approximations and a classification of the singularities you can get in their quotient schemes.

So if you are interested in this or want to have the details of the theorems i mentioned before you can download it from this short URL.

14 : But what does all of this has to do with our algebraic D-branes? Recall that our naive algebraic view of a stack of n D-branes was to view them as an algebra map from the coordinate ring of your favourite space-time variety to an Azumaya algebra of degree n over the locus of the branes. Or we can define more generally an algebraic D-brane in a possibly noncommutative algebra R over a commutative ring C to be an algebra map from R to an Azumaya algebra over C.

We have seen that we can associate to this map an equivariant map between the n-th representation schemes and this data contains enough information to reconstruct the original map by going to the rings of equivariant maps to $n \times n$ complex matrices. By Artin's result we know that this algebra coincides with the Azumaya algebra and for R it only gives us the level n approximation but we can then use again our universal map j_n to recover the algebra map from R to A.

In the special case considered by Liu and Yau we now know what the noncommutative space-time should be that allows us to embed all n stacks of D-branes, namely the prime spectrum of the level n-approximation of the coordinate ring of Y. In the simple example of the affine line we get the polynomial ring in n variables with x corresponding to the complex matrix and the x_i being teh trace of the i-th power of this matrix. So we get indeed that in this case the space should be n-dimensional but we are now able to compute these spaces moire generally.

15 : Recall that geometric invariant theory enabled us only to parametrize the closed orbits by the points of the quotient scheme but one would love to describe all orbits in a geometric way. Artin, Mumford and Deligne showed that one can do this if allow more general geometric objects than schemes namely algebraic stacks. So what are these?

Recall that an affine scheme was a functor from commutative algebras to sets, a stack will be a functor from commutative algebras to groupoids, where a groupoid is a category in which every morphism is an isomorphism.

Note that we can turn every set into a groupoid by taking as the only morphisms an identity map for each element of the set. In this way all affine schemes are special instances of algebraic stacks.

In particular, they defined the quotient stack as the functor that associates to a commutative algebra C the collection of all settings where Y is a principal G-bundle over the prime spectrum together with a G-equivariant map to X. Morphisms between two such settings are given by equivariant maps making the diagram commute and as all G-equivariant maps between principal G-bundles can be inverted this is indeed a groupoid.

What are the complex points of this functor. Well, over $\mathbb{C}$ there is just 1 principal G-bundle namely the group G itself and a G-equivariant map sends the unit to a point in X and its image is the G-orbit of that point. Maps between two such complex points only alter the given point in the orbit so isoclasses are really the G-orbits in X.

Nice so we have a functor describing all G-orbits but what is the geometry of this so called quotient stack and in particular can we make sense of the quotient-map sending a point to its orbit?

16 : Well we can probe our mystery object by affine schemes. That is we can describe maps from $\mathtt{spec}(C)$ to the quotient stack. Because both are functors the correct definition of a map between them should be a natural transformation and an old idea of Yoneda identifies such natural transformations as the C-points of the quotient-stack. The correspondence is easy, C-points of $spec(C)$ are the algebra maps from C to itself and in this set there is a special element namely the identity map and this is send under a natural transformation to a C-point of the quotient stack. What Yoneda proved was that this element determines the transformation.

In particular, the elusive quotient map π is determined by a principal G-bundle over X with an equivariant map to X. There is a canonical choice, namely the trivial G-bundle together with the action map to X.

An important property of the map π is that it is representable. By this we mean that if you take any prime spectrum and any map α and form the stacky fiber-product (the details of which i'll spare you) then this fiber-product is actually an affine scheme and one easily verifies that the required scheme is just the principal G-bundle corresponding to the map α.

This allows us to define properties of the quotient-map as the collective properties of all the maps π_α which are now maps between ordinary schemes.

If G is a finite group then all the π_α are etale so in this case the quotient map is etale and one calls the corresponding stack a Deligne-Mumford stack whereas if G is a reductive group then alle π_α are smooth and so the quotient map is smooth and we call it an Artin stack.

In particular if we have a G-action on a smooth variety X then the quotient-stack not only parametrizes the orbits but it is a smooth object in the stacky world.

17 : After this short introduction to algebraic stacks we will now describe the C-points of the quotient representation stack. By defnition they correspond to principal PGL_n-bundles over the prime spectrum together with an equivariant map to the n-th representation scheme of R. By Artin's result we know that these principal bundles are exactly the representation schemes of Azumaya algebras of degree n.

Again we can take rings of equivariant maps on both sides and compose it with the universal map to get an algebra map from R to the Azumaya algebra, that is, we have identified the C-points of the quotient representation stack as the n-stacks of algebraic branes in R over C and the different use of stack in both cases makes it somewhat confusing.

In particular, if we start with a formally smooth algebra R then for every n these quotient stacks are smooth, so one should expect a good behavior for algebraic branes in this case.

18 : I'd like to close with two suggestions for further work. In string theory D-branes are not stationary but rather dynamic objects, so branes can merge or separate from the stack and in our naive algebraic framework this corresponds to the deformation of the algebra maps.

We define one algebraic brane, that is an algebra map f to deform to another if there are more elements in A centralizing the image of f than centralizing the image of g. The idea behind this definition is that these centralizer algebras should be viewed as a version of the Lie algebras of stabilizer groups and deformation should correspond to symmetry breaking.

In order to study these deformations geometrically one can define the A-representation scheme of R by mimicking Bergman's old idea which works just as well because A is an Azumaya algebra and we can use the centralizer result of maps from Azumaya algebras to show that the abelianization of the A-th root of R then represents this functor of which the C-points are exactly the algebraic branes from R to A and one can even define an A-version of the representation stack.

There is one difference though. The A-representation scheme is not an affine scheme over the complex numbers but rather over $\texttt{spec}(C)$.

19 : Finally, the algebraic framework i've described raises a question about the process of stacking more and more D-branes on the same locus. In our setting a larger stack of algebraic branes would mean a map to an Azumaya algebra of larger degree over C. But one should connect these two stacks of branes somehow and the idea must be that the bigger stack contains more information than the smaller one.

A possibility to define this relationship between the two branes might be that there should be a morphism between the two Azumaya algebras such that the bigger map is a deformation from the composition of the smaller with the Azumaya-map.

A simple illustration of a family of algebraic branes relatedin this way is given here using teh quantum-tori stacked over the maximal torus of GL_2. I think such families of maps to Azumaya algebras deserve a closer attention.

Anyway, let's return to the compatibility map between the small and bigger Azumaya algebra. By the double-centralizer result such maps can arise only if the degree of the smaller Azumaya is a divisor of that of the bigger one.

This suggests that the stacking process of D-branes has a multiplicative aspect to it rather than an additive one and one of the things i'd love to find out over this week is whether this rings a bell to some of the physicists present. Thanks!

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