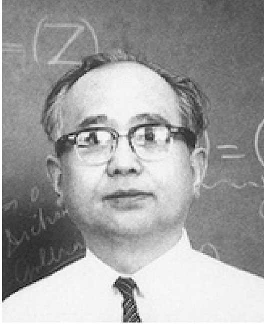
algebraic D-branes

leiden, march 2012

Goro Azumaya (1951)



A Azumaya algebra with center C

- $A^e = A \otimes_C A^{op} \to End_C(A)$ iso
- $\forall \mathfrak{m} \in \mathtt{max}(C) \;:\; \widehat{A_\mathfrak{m}} \simeq M_n(\widehat{C_\mathfrak{m}})$

Brauer group $Br(C)$

- Azumayas closed under $\otimes_C$
- $Br(C) =$ Morita-classes of Azumayas over C
- $Br(C) = H^2_{et}(\mathtt{spec}(C), \mathbb{G}_m)_{torsion}$

- $\mathtt{mod}(C) \equiv \mathtt{bimod}(A)$
- $A \to R$ a C-morphism, then $R = A \otimes_C R^A$

(quantum) 2-torus

$X = \mathbb{C}^* \times \mathbb{C}^*$

$C = \mathcal{O}(X) = \mathbb{C}[s^{\pm 1}, t^{\pm 1}]$

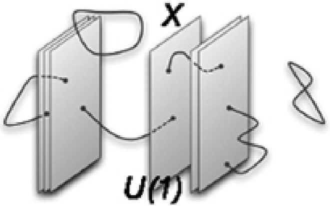
$A_n = \mathbb{C}_{\zeta_n}[U_n^{\pm 1}, V_n^{\pm 1}]$, $V_n U_n = \zeta_n U_n V_n$, $\zeta_n^n = 1$

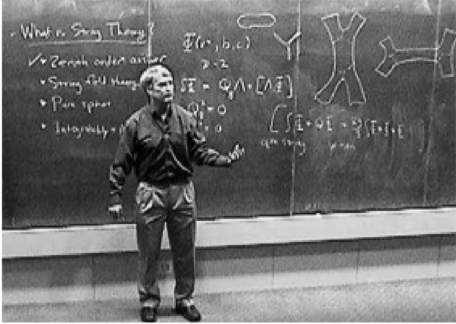
$Z(A_n) = \mathbb{C}[U_n^{\pm n}, V_n^{\pm n}] = \mathbb{C}[s^{\pm 1}, t^{\pm 1}] = C$

A_n is Azumaya of degree n over C

$Br(C) = Br(X) = \mathbb{Q}/\mathbb{Z}$

Joe Polchinski (1989)

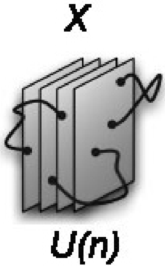




one D-brane on X : $\mathbb{C}[Y] \twoheadrightarrow \mathbb{C}[X]$

n-stack of D-branes on X : $\mathbb{C}[Y] \longrightarrow A$

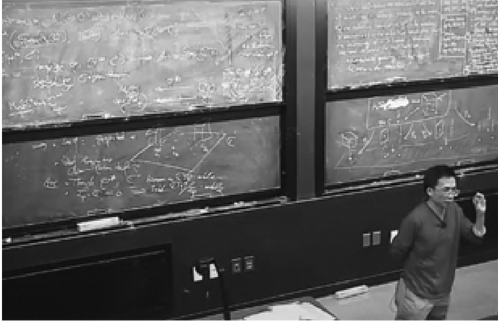
$$\begin{array}{c} A \\ \uparrow \scriptstyle{\mathtt{Azu}_n} \\ \mathbb{C}[X] \end{array}$$



C-H. Liu, S-T. Yau (2007)

$$\begin{array}{ccc} \texttt{nc-space}(A) & \dashrightarrow & Y_{(\texttt{nc})} \\ \downarrow \approx & & \\ X & & \end{array}$$

"Azumaya nc-geometry"



0709.1515,0809.2121,0901.0342,0907.0268,0909.2291,1003.1178,1012.0525,1111.4707,...

problem : most nc-space proposals are not functorial

$$\begin{array}{ccc} \mathbb{C}[x] & \longrightarrow & M_n(\mathbb{C}) \\ & & \uparrow \\ & & \mathbb{C} \end{array} \qquad \begin{array}{ccc} \bullet & \overset{?}{\dashrightarrow} & \mathbb{A}^1 \\ \downarrow & & \\ \bullet & & \end{array} \qquad \begin{array}{ccc} \bullet & \dashrightarrow & \mathbb{A}^1_{\texttt{nc}} \approx \texttt{Hilb}_n(\mathbb{A}^1) \\ \downarrow & & \\ \bullet & & \end{array}$$

NAG & functoriality

- $R \longrightarrow S$
- $\texttt{rep}(R) \longleftarrow \texttt{rep}(S)$

category theory?

- $\texttt{rep}(R) = \underset{\rightarrow}{lim}\ \texttt{rep}_n(R)$
- Artin-Procesi (1969) : study $\texttt{rep}_n(R)$ via GIT

PGL_n-equivariant geometry?

- Kontsevich (1999) : nc-geometric gadgets of R induces equivariant ones on *all* the $\texttt{rep}_n(R)$

George Bergman (1973)

$$\sqrt[n]{R} = (R * M_n(\mathbb{C}))^{M_n(\mathbb{C})}$$

$$\Downarrow$$

$$M_n(\sqrt[n]{R}) = R * M_n(\mathbb{C})$$

<u>noncommutative representation scheme</u>

$\sqrt[n]{R}$ represents the functor

$$\mathtt{alg} \to \mathtt{sets} \qquad B \mapsto Alg(R, M_n(B))$$

that is, $Alg(R, M_n(B)) = Alg(\sqrt[n]{R}, B)$

Alexander Grothendieck (1958)

$X \ : \ \texttt{commalg} \rightarrow \texttt{sets}$

is affine iff $\forall C \in \texttt{commalg}$

$X(C) = Alg(\mathbb{C}[X], C)$

representation scheme

$\sqrt[n]{R}_{ab} = \sqrt[n]{R}/[\sqrt[n]{R}, \sqrt[n]{R}]$ represents the functor

$\texttt{rep}_n(R) \ : \ \texttt{commalg} \rightarrow \texttt{sets} \qquad C \mapsto Alg(R, M_n(C))$

that is, $\mathbb{C}[\texttt{rep}_n(R)] = \sqrt[n]{R}_{ab}$

universal map

$$\begin{array}{ccc} R & \longrightarrow & R * M_n(\mathbb{C}) \\ {\scriptstyle j_n}\downarrow & & \downarrow{\scriptstyle =} \\ M_n(\mathbb{C}[\mathtt{rep}_n(R)]) & \longleftarrow & M_n(\sqrt[n]{R}) \end{array}$$

PGL_n-action

$$\begin{array}{ccc} R & \longrightarrow & R * M_n(\mathbb{C}) \\ & {\scriptstyle \phi_g}\searrow & \downarrow{\scriptstyle id * c_g} \\ & & R * M_n(\mathbb{C}) \end{array} \qquad \begin{array}{c} \sqrt[n]{R} \\ \downarrow{\scriptstyle \psi_g} \\ \sqrt[n]{R} \end{array} \qquad \begin{array}{c} \mathbb{C}[\mathtt{rep}_n(R)] \\ \downarrow{\scriptstyle \overline{\psi_g}} \\ \mathbb{C}[\mathtt{rep}_n(R)] \end{array}$$

'generalized' representation schemes

$$\sqrt[n]{R}_{ab} = ((R * M_n(\mathbb{C}))^{M_n(\mathbb{C})})_{ab}$$

Mike Artin (1969)

$$\begin{array}{c} \mathtt{rep}_n(R) \\ \pi \downarrow GIT \\ \mathtt{rep}_n(R)/PGL_n \end{array} = \mathtt{iss}_n(R)$$

- principal PGL_n-fibrations over $\mathtt{spec}(C)$
- $\mathtt{rep}_n(A) \twoheadrightarrow \mathtt{iss}_n(A) = \mathtt{spec}(C)$, A/C $\mathtt{Azu}_n$
- $A = \int_n A = \{ \mathtt{rep}_n(A) \xrightarrow{\mathtt{equiv}} M_n(\mathbb{C}) \}$
- $C = \oint_n A = \mathbb{C}[\mathtt{rep}_n(A)]^{PGL_n}$

Michael Artin, On Azumaya algebras and finite dimensional representations of rings, J. Algebra 11 (1969)

Claudio Procesi (1976)

$$\begin{array}{ccc} \int_n R : \mathtt{rep}_n(R) & \longrightarrow & M_n(\mathbb{C}) \\ \pi \downarrow GIT & & \\ \oint_n R : \mathtt{iss}_n(R) & & \end{array}$$

$$Sym(R/[R,R]_v) \xrightarrow{tr=Tr(j_n)} \oint_n R = \mathbb{C}[\mathtt{rep}_n(R)]^{PGL_n}$$

$$R \otimes Sym(R/[R,R]_v) \xrightarrow{j_n \otimes tr} \int_n R = M_n(\mathbb{C}[\mathtt{rep}_n(R)])^{PGL_n}$$

Claudio Procesi, The invariant theory of $n \times n$ matrices, Advances in Math. 19 (1976)

Maxim Kontsevich (1999)

$$\begin{array}{c} \texttt{nc-geometry}(R) \\ MM_n \big\downarrow \forall n \\ \texttt{equiv-geo}\ \texttt{rep}_n(R) \end{array}$$

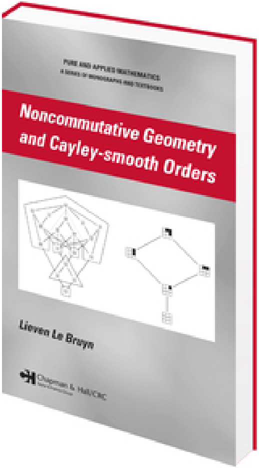
$$R \xrightarrow{j_n} M_n(\mathbb{C}[\texttt{rep}_n(R)])$$

$$\begin{array}{c} M_n(\mathbb{C}[\texttt{rep}_n(R)]) \\ \big\uparrow \\ \mathbb{C}[\texttt{rep}_n(R)] \end{array}$$

$$\begin{array}{c} \texttt{bimod}(R) \\ MM_n \big\downarrow - \otimes_{R^e} M_n(\mathbb{C}[\texttt{rep}_n(R)]) \\ \texttt{mod}(\mathbb{C}[\texttt{rep}_n(R)]) \end{array}$$

Maxim Kontsevich, Non-commutative smooth spaces, Arbeitstagung (1999)
Michel Van den Bergh, Non-commutative quasi-Hamiltonian spaces, Contemp. Math. (2008)

commercial break



R formally smooth

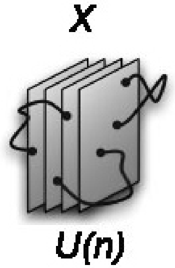
$$\Downarrow$$

$\mathtt{rep}_n(R)$ is smooth, $\forall n$

- étale local structure of $\int_n R$
- étale local structure of $\mathtt{rep}_n(R)/PGL_n$

bit.ly/ySULGZ

n-stack of algebraic branes in R over C



$$\begin{array}{ccc} \mathbb{C}[Y] & \longrightarrow & A \\ & & \uparrow{\scriptstyle \mathtt{Azu}_n} \\ & & \mathbb{C}[X] \end{array} \qquad \begin{array}{ccc} R & \xrightarrow{f} & A \\ & & \uparrow{\scriptstyle \mathtt{Azu}_n} \\ & & C \end{array}$$

$$\begin{array}{ccc} \mathtt{rep}_n(R) & \xleftarrow{f^*} & \mathtt{rep}_n(A) \\ & & \downarrow \\ & & \mathtt{spec}(A) \end{array} \qquad \begin{array}{ccc} \int_n R & \longrightarrow & \int_n A = A \\ \uparrow{\scriptstyle j_n} & \overset{f}{\nearrow} & \uparrow \\ R & & C \end{array}$$

$$Y_{\mathtt{nc}} = \mathtt{spec}(\textstyle\int_n \mathbb{C}[Y]) \qquad \int_n \mathbb{C}[x] = \mathbb{C}[x, x_1, \ldots, x_{n-1}]$$

David Mumford (1969)

$$\begin{array}{c} X \\ \downarrow{\scriptstyle \pi} \\ X/G \end{array} \qquad\qquad \begin{array}{c} X \\ \downarrow{\scriptstyle \pi} \\ [X/G] \end{array}$$

$$[X/G] \ : \ \texttt{2-commalg} \to \texttt{groupoids}$$

$$\begin{array}{ccc} Y & \xrightarrow{G-equiv} & X \\ {\scriptstyle G-principal}\downarrow & & \\ \mathtt{spec}(C) & & \end{array} \qquad\qquad \begin{array}{ccc} Y & \longrightarrow & X \\ \downarrow & \overset{G-equiv}{\searrow} & \uparrow \\ \mathtt{spec}(C) & \longleftarrow & Y' \end{array}$$

Nobuo Yoneda (1954)

$$\texttt{2-commalg} \underset{[X/G]}{\overset{\texttt{spec}(C)}{\rightrightarrows}} \texttt{groupoids} \qquad \downarrow N$$

$$\{\, \texttt{spec}(C) \xrightarrow{N} [X/G] \,\} = [X/G](C)$$

$$\begin{array}{ccccc}
X & & G \times X \xrightarrow{\texttt{act}} X & & Y_\alpha = \texttt{spec}(C) \times_{[X/G]} X \longrightarrow X \\
\downarrow \pi & & \downarrow & & \pi_\alpha \downarrow \qquad\qquad\qquad \downarrow \pi \\
[X/G] & & X & & \texttt{spec}(C) \xrightarrow{\alpha} [X/G]
\end{array}$$

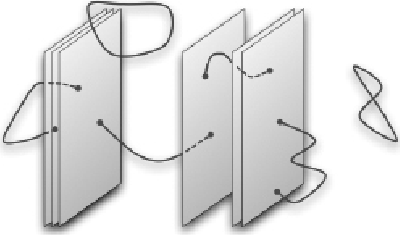
- G finite $\implies$ π étale (Deligne-Mumford stack)
- G reductive $\implies$ π smooth (Artin stack)

(quotient) representation stack $[\mathtt{rep}_n(R)/PGL_n]$

$$\begin{array}{ccc} \mathtt{rep}_n(A) & \xrightarrow{equiv} & \mathtt{rep}_n(R) \\ \big\downarrow{\scriptstyle Azu_n} & & \\ \mathtt{spec}(C) & & \end{array} \qquad \begin{array}{ccc} A = \int_n A & \longleftarrow & \int_n R \\ \big\uparrow & & \big\uparrow{\scriptstyle j_n} \\ C & & R \end{array}$$

- C-points in stack $[\mathtt{rep}_n(R)/PGL_n]$
- n-stack of algebraic branes in R over C

R formally smooth $\implies \forall n \; : \; [\mathtt{rep}(R)/PGL_n]$ smooth dynamic aspect = deformation



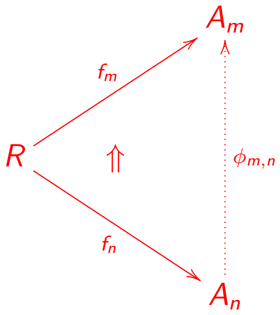
$$R \underset{g}{\overset{f}{\rightrightarrows}} A \quad \Downarrow\alpha \qquad A^{Im(f)} \supset A^{Im(g)}$$

$\sqrt[A]{R}_{ab} = ((R *_C A)^A)_{ab}$ represents the functor

$$\mathtt{rep}_A(R) \;:\; \mathtt{commalg}_C \to \mathtt{sets} \quad B \mapsto Alg(R, A \otimes_C B)$$

$Aut_C(A)$ acts on $\mathtt{rep}_A(R) \implies [\mathtt{rep}_A(R)/Aut_C(A)]$

families of algebraic branes



$$A_m = A_n \otimes_C \underbrace{A_m^{A_n}}_{\mathtt{Azu}_{\frac{m}{n}}} \implies n|m$$

quantum tori

$$\phi_{m,n} : \mathbb{C}_{\zeta_n}[U_n^{\pm 1}, V_n^{\pm 1}] \to \mathbb{C}_{\zeta_m}[U_m^{\pm 1}, V_m^{\pm 1}] \qquad U_n \mapsto U_m^{\frac{m}{n}}, V_n \mapsto V_m^{\frac{m}{n}}$$

$$GL_2 = \begin{bmatrix} s & u \\ v & t \end{bmatrix} \qquad f_n \; : \; \mathbb{C}[GL_2] \to \mathbb{C}_{\zeta_n}[U_n^{\pm 1}, V_n^{\pm 1}] \quad \begin{cases} u \mapsto 0 \\ v \mapsto 0 \\ s \mapsto s = U_n^n \\ t \mapsto V_n \end{cases}$$